UNIT - 1

Ordinary Differential equations of First order:

Differential equation: An eqⁿ involving one dependent variable and its derivative w.r.t one or more independent variables is known as a differential equation.

⇒ Differential equations are of two types

i) Ordinary Differential equation.

ii) Partial Differential equation.

Ordinary Differential eqⁿ: A D.E involving one dependent variable and its derivative w.r.t one independent variable is known as ordinary differential equation.

Ex: $\frac{dy}{dx} + y = e^x$; $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Partial Differential equation: A D.E involving one dependent variable and its derivative w.r.t two or more independent variables is known as partial D.E denoted by '∂'.

Ex: $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$; $\frac{\partial^2 z}{\partial x \cdot \partial y} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$

Degree of Differential equation: The degree of a differential eqⁿ is defined as the power to which the highest order derivative is raised.

Ex: $\frac{d^4y}{dx^4} + \left(\frac{d^2y}{dx^2}\right)^2 - 3\frac{dy}{dx} + y = 9 \Rightarrow$ degree $= 1$

Order of Differential eqⁿ: The order of a differential eqⁿ is defined to be that of the highest order derivative it contains.

Ordinary Differential eq^n of first order:

The differential eq^n of the form $\frac{dy}{dx} = f(x,y)$ is known as D.E of first order, which are classified as follows.

a) Variable separable form.

b) Homogeneous Differential equation.

c) Non-Homogeneous differential equation.

d) Exact differential equation.

e) Non-Exact differential eq^n (reducable to exact)

f) Linear Differential equation.

g) Bernoulli's Differential equation.

Variable seperable form: The first order differential eq^n of the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ is known as Variable seperable which on integration gives the solution (i.e., x variables can be seperated from y variables).

ex: ① $\frac{dx}{dy} = \frac{x}{y}$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c$$

② $\frac{dy}{dx} = e^{x+y}$

$$\frac{dy}{dx} = e^{x} \cdot e^{y}$$

$$\frac{dy}{e^{y}} = e^{x} \cdot dx$$

$$\int e^{-y} \cdot dy = \int e^{x} dx$$

$$-e^{-y} = e^{x}$$

$$e^{x} + e^{-y} = e^{c}$$

* $\frac{dy}{dx} + \frac{\sqrt{1+y^2}}{\sqrt{1+x^2}} = 0$

$$\frac{dy}{dx} = -\sqrt{\frac{1+y^2}{\sqrt{1+x^2}}}$$

$$\frac{dy}{\sqrt{1+y^2}} = \frac{-dx}{\sqrt{1+x^2}}$$

$$\int \frac{dy}{\sqrt{1+y^2}} = -\int \frac{dx}{\sqrt{1+x^2}}$$

$$\sinh^{-1}(y) = -\sinh^{-1}(x)$$

$$\sinh^{-1}(y) + \sinh^{-1}(x) = \sinh^{-1}c.$$

Homogeneous D.E:

The first order d.e $\frac{dy}{dx} = f(x,y)$ is said to be homogeneous D.E if $f(x,y)$ is homogeneous function of degree zero which can be solved by substituting and reduces to variable seperable form.

$$y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Ex: ① $f(x,y) = \frac{x^2+y^2}{x-y}$

$$= \frac{k^2x^2+k^2y^2}{kx-ky} = \frac{k^2[x^2+y^2]}{k[x-y]}$$

$$= k f(x,y)$$

It is Homogeneous of degree 1.

② $f(x,y) = \frac{kx+ky}{kx-ky}$

$$= \frac{\cancel{k}(x+y)}{\cancel{k}(x-y)}$$

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$= f(x,y)$

It is a homogeneous of degree 0.

Q) solve $[x^2y - 2xy^2]dx = [x^3 - 3x^2y]dy$

$$\frac{x^2y-2xy^2}{x^3-3x^2y} = \frac{dy}{dx}$$

$$f(x,y) = \frac{k^2x^2ky - 2kx\,k^2y^2}{k^3x^3 - 3k^2x^2\,ky}$$

$$= \frac{\cancel{k^3}(x^2y-2xy)}{\cancel{k^3}(x^3-3x^2y)}$$

It is homogeneous of degree '0'.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^3v - 2x^3v}{x^3 - 3x^3v} = \frac{-v}{1-3v}$$

$$x\frac{dv}{dx} = \frac{-v}{1-3v} - v = \frac{-v-v+3v^2}{1-3v}$$

$$x\frac{dv}{dx} = \frac{3v^2-2v}{1-3v}$$

$$\int \frac{1-3v}{3v^2-2v}\,dv = \int \frac{dx}{x}$$

Put $3v^2 - 2v = t$

$(6v-2)\,dv = dt$

$-2(1-3v)\,dv = dt$

$(1-3v)\,dv = \frac{dt}{-2}$

$$-\frac{1}{2}\int\frac{dt}{t} = \int\frac{dx}{x}$$

$-\frac{1}{2}\log t = \log x + \log c$

$\log x + 2\log t = \log c$

$x + t^2 = c$

$\Rightarrow x(3v^2-2v)^2 = c$

$x\left[\frac{3y^2}{x^2} - \frac{2y}{x}\right]^2 = c$

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Non-Homogeneous D.E:

The D.E of form $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is known as non-homogeneous D.E.

(case i) if $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}}$

then $\frac{dy}{dx} = \frac{k(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2}$ which can be solved by taking

$z = a_2x + b_2y$

(case ii) if $\boxed{\frac{a_1}{a_2} \neq \frac{b_1}{b_2}}$ then put $x = X+h \Rightarrow dx = dX$

$y = Y+k \Rightarrow dy = dY$

$\Rightarrow a_1h + b_1k + c_1 = 0$ and

$a_2h + b_2k + c_2 = 0$

(case iii) if $\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$ then it directly reduces to variable seperable form.

Q) solve $\frac{y+x-2}{y-x-4} = \frac{dy}{dx}$

$a_1 = 1,\ b_1 = 1,\ c_1 = -2$

$a_2 = -1,\ b_2 = 1,\ c_2 = -4$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

Put $x = X+h \Rightarrow dx = dX$
$y = Y+k \qquad dy = dY$

$\Rightarrow a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$

$h+k-2=0 \qquad -h+k-4=0$

on solving

$h = -1,\ k = 3$

	h		k		2	
1		-2		1		1
1		-4		-1		1

$$\frac{h}{-4+2} = \frac{k}{2+4} = \frac{1}{1+1} \Rightarrow \frac{h}{-2} = \frac{k}{6} = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{(y+k)+(x+h)-2}{(y+k)-(x+h)-4} \qquad h=-1, \quad k=3$$

$$= \frac{y+k+x+h-2}{y+k-x-h-4} = \frac{y+3+x-1-2}{y+3-x+1-4}$$

$$= \frac{y+x}{y-x}$$

which is a homogeneous D.E and can be solved by taking $y = vx$

$$v + x\frac{dv}{dx} = \frac{vx+x}{vx-x} = \frac{v+1}{v-1}$$

$$x\frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{v+1-v^2+v}{v-1}$$

$$= \frac{-v^2+2v+1}{v-1}$$

$$\int \frac{v-1}{-v^2+2v+1}\, dv = \int \frac{dx}{x}$$

$$= \frac{-1}{2}\int \frac{-2v+2}{-v^2+2v+1}\, dv = \int \frac{dx}{x}$$

$$-\frac{1}{2}\log(-v^2+2v+1) = \log x + \log c$$

$$2\log x + \log(-v^2+2v+1) = \log c$$

$$x^2 \cdot (1-2v-v^2) = c$$

$$x^2\left[1 + \frac{2v}{x} - \frac{v^2}{x^2}\right] = c$$

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Exact differential equations:

The first order D.E $Mdx + Ndy = 0$, where m, n are functions of x & y is said to be exact. if their exist a non zero function.

$f(x,y)$ such that $M = \frac{\partial f}{\partial x}$; $N = \frac{\partial f}{\partial y}$

Condition for exactness.

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then given D.E is exact and the solution is given by $\underset{(y\ const)}{\int M dx} + \int (\text{terms independent of } x \text{ in } N) dy = c.$

Q) solve $(2x-y+1)dx + (2y-x-1)dy = 0$

$M = 2x-y+1$ $\qquad$ $N = 2y-x-1$

$\frac{\partial M}{\partial y} = -1$ $\qquad$ $\frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -1$

$\Rightarrow$ Given D.E is exact and sol^n is given by

$\underset{(y\ const)}{\int M dx} + \int (\text{terms independent of } x \text{ in } N) dy = c$

$\int (2x-y+1) dx + \int (2y-1) dy = c$

$\int 2x\,dx - y\int 2dx + \int dx + \int 2y\,dy - \int dy = c$

$x^2 - yx + x + y^2 - y = c$

$x^2 + y^2 + x - y - yx = c.$

Q) $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

$\frac{dy}{dx} = -\frac{(y\cos x + \sin y + y)}{\sin x + x\cos y + x}$

$0 = +(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x) dy$

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$M = +(y\cos x + \sin y + y)$ $\quad N = \sin x + x\cos y + 1$

$\frac{\partial M}{\partial y} = + y\cos x + \sin y + y$ $\quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$= \cos x + \cos y + 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$\int M\,dx + \int (\text{terms independent of } x \text{ in } N)\,dy = c$

y const

$\int (y\cos x + \sin y + y)dx + \int 0 = c$

$y\cdot(\sin x) + \sin y \cdot x + yx = c$

$y\sin x + x\sin y + xy = c$

Non-exact of D.E reducable to exact:

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then given D.E.

$M dx + N dy = 0$ is said to be non exact which can be reducable to exact when multiplied by a non zero function $f(x,y)$ known as <u>integrating factor</u>.

Methods to find integrating factor.

Method -1:

1) $d(xy) = x\,dy + y\,dx$

2) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

3) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

4) $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$

5) $d\left[\log\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{xy}$

6) $d\left[\log\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{xy}$

7) $d\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{y\,dx - x\,dy}{x^2+y^2}$

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8) $d\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{x\,dy - y\,dx}{x^2+y^2}$

9) $d\left[\log(xy)\right] = \frac{y\,dx + x\,dy}{xy}$

10) $d\log(x^2+y^2) = \frac{2(x\,dx + y\,dy)}{x^2+y^2}$

11) $d\left(\frac{e^x}{y}\right) = \frac{y e^x dx - e^x dy}{y^2}$

Method 2:

If $Mx + Ndy = 0$ is the first order D.E $\ni M, N$ are homogeneous of same degree and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is integrating factor.

Method-3:

If the given D.E in form $y f(xy)dx + x g(xy)dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is integrating factor.

Method-4:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ then $e^{\int f(x)dx}$ is an integrating factor.

Method-5:

If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$ then $e^{-\int g(y)dy}$ is an integrating factor.

Q) solve $y[x^2y^2+2]dx + x[2-2x^2y^2]dy = 0$ —① :

Sol. $M = y(x^2y^2+2)$ $\qquad N = x(2-2x^2y^2)$

$= x^2y^3+2y$ $\qquad = 2x - 2x^3y^2$

$\frac{\partial M}{\partial y} = 3x^2y^2+2$ $\qquad \frac{\partial N}{\partial x} = 2 - 6x^2y^2$

As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and also given DE is in the form

$yf(xy)dx + xg(xy)dy = 0$

$Mx - Ny = x^3y^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3$

$= 3x^3y^3 \neq 0$

$\frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$ is I.F

$$\frac{\cancel{y}[x^2y^2+2]dx}{3x^3y^{\cancel{3}}} + \frac{\cancel{x}(2-2x^2y^2)dy}{3x^{\cancel{3}}y^3} = 0$$

$$\frac{x^2y^2+2}{3x^3y^2}dx + \frac{2-2x^2y^2}{3x^2y^3}dy = 0$$

$$\left[\frac{\cancel{x^2}\cancel{y^2}}{3x^{\cancel{3}}\cancel{y^2}} + \frac{2}{3x^3y^2}\right]dx + \left[\frac{2}{3x^2y^3} - \frac{2\cancel{x^2}\cancel{y^2}}{3\cancel{x^2}y^{\cancel{3}}}\right]dy = 0$$

$$\left[\frac{1}{3x} + \frac{2}{3x^3y^2}\right]dx + \left[\frac{2}{3x^2y^3} - \frac{2}{3y}\right]dy = 0 \quad \text{—②}$$

$M_1 = \frac{1}{3x} + \frac{2}{3x^3y^2}$ $\qquad N_1 = \frac{2}{3x^2y^3} - \frac{2}{3y}$

eqn is in the form $M_1dx + N_1dy = 0$

where $M_1 = \frac{1}{3x} + \frac{2}{3x^3y^2}$. $\qquad N_1 = \frac{2}{3x^2y^3} - \frac{2}{3y}$

$\frac{\partial M_1}{\partial y} = 0 + \frac{2}{3x^3}\left(\frac{-2}{y^3}\right)$ $\qquad \frac{\partial N_1}{\partial x} = \frac{2}{3y^3}\left(\frac{-2}{x^3}\right)$

$= \frac{-4}{3x^3y^3}$ $\qquad = \frac{-4}{3x^3y^3}$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\Rightarrow$ The given eq^n is exact and sol^n is given by

$\int M_1 dx + \int$ (terms independent of x in N) $dy = c$
(y const)

$$\int \frac{1}{3x} + \frac{2}{3x^2y^2}\,dx + \int \frac{-2}{3y}\,dy = c$$

$$\frac{1}{3}\int \frac{1}{x}\,dx + \frac{2}{3}\int \frac{1}{x^3}\,dx - \frac{2}{3}\int \frac{1}{y}\,dy = c$$

$$\frac{1}{3}\log x + \frac{\cancel{2}}{3y^2}\cdot\frac{x^{-2}}{\cancel{2}} - \frac{2}{3}\log y = c$$

$$\frac{1}{3}\log x - \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$$

$$\log x - \frac{1}{x^2y^2} - 2\log y = c$$

$$\log\frac{x}{y^2} - \frac{1}{x^2y^2} = c$$

Q) solve $y(2xy + e^x)\,dx - e^x\,dy = 0$

$M = 2xy^2 + ye^x \qquad N = -e^x$

$\frac{\partial M}{\partial y} = 4xy + e^x \qquad \frac{\partial N}{\partial x} = -e^x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy + e^x + e^x = 4xy + 2e^x$$
$$= 2(2xy + e^x)$$

$$\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{2\cancel{(2xy+e^x)}}{y\cancel{(2xy+e^x)}} = \frac{2}{y}$$

I.F $e^{-\int f(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y}$
$$= y^{-2}$$

$$\frac{1}{y^2}\left[y\cdot(2xy + e^x)\,dx\right] - \frac{1}{y^2}e^x\,dy = 0$$

$$\left[\frac{2(xy + e^x)}{y}\right]dx - \frac{e^x}{y^2}\,dy = 0$$

$$2\left[x + \frac{e^x}{y}\right]dx - \frac{e^x}{y^2}\,dy = 0$$

$M_1 = 2\left[x + \frac{e^x}{y}\right]$ $\quad N_1 = -\frac{e^x}{y^2}$

$\frac{\partial M_1}{\partial y} = 0 - \frac{e^x}{y^2}$ $\quad = -\frac{e^x}{y^2}$

$= -\frac{e^x}{y^2}$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\int\limits_{(y\,const)} M_1\,dx + \int (\text{terms independent of } x \text{ in } N)\,dy = c$

$\int\limits_{(y\,const)} \left(2x + \frac{2e^x}{y}\right) dx + \int 0 = c$

$\int x\,dx + \int \frac{e^x}{y}\,dx$

$\frac{x^2}{2} + \frac{e^x}{y} = c$

Linear Differential equation:

The DE of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is known as linear D.E in y of 1^{st} order whose integrating factor is $e^{\int P(x)\,dx}$ and sol.^{n} is given by $y(I.F) = \int Q\,(I.F)\,dx + c$.

→ Similarly D.E of the form

$\frac{dx}{dy} + P(y)\,x = Q(y)$ is known as linear DE in x whose I.F is $e^{\int P(y)dy}$ and sol.^{n} is given by

$x\,(I.F) = \int Q\,(I.F)\,dy + c$.

NOTE:

→ A DE is said to be linear if

1) It contains only y' (should not contain any other power of y).

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2) It should not contain $y\frac{dy}{dx}$

3) should not contain trigonometric (logarithmic) exponential in y.

* similarly for x.

Q) $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

$\frac{dy}{dx} + \frac{1}{x\log x}\, y = \frac{\sin 2x}{\log x}$

It is linear in y

$\frac{dy}{dx} + P(x)\cdot y = Q(x)$

$P(x) = \frac{1}{x\log x}$ $\qquad Q(x) = \frac{\sin 2x}{\log x}$

I.F is $e^{\int P(x)\cdot dx} = e^{\int \frac{1}{x\log x}\cdot dx}$

put $\log x = t$

$\frac{1}{x} = \frac{dt}{dx}$

$\frac{dx}{x} = dt$

$I.F = e^{\int \frac{dt}{t}} = e^{\int \log t} = t = \log x$

solution is given by $\Rightarrow y\cdot I.F = \int Q\cdot I.F\, dx + c$

$= \int \frac{\sin 2x}{\cancel{\log x}}\cdot \cancel{\log x}\, dx = \frac{-\cos 2x}{2}$

$\sin x = t$

$\cos x\, dx = dt$

$= \int 2\sin x \cos x\, dx + c$

$= \int 2t\, dt$

$= \cancel{2}\left[\frac{t^2}{\cancel{2}}\right]$

$= \sin^2 x + c$

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Bernoullis equation:

An eqn of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is known as bernoulli's eqn which can be reduced to linear D.E as follows.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

Put $y^{1-n} = z$

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)\cdot z = Q(x)$$

which is a linear D.E in z.

Q) solve $x\frac{dy}{dx} + y = x^3y^6$

sol. $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$$

Put $y^{-5} = z$

$$-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dz}{dx}$$

$$-\frac{1}{5}\frac{dz}{dx} + \frac{y^{-5}}{x} = x^2$$

$$\frac{dz}{dx} - \frac{5}{x}z = -5x^2$$

which is clearly linear in z.

$P(x) = \frac{-5}{x}$, $Q(x) = -5x^2$

I.F $e^{-\int\frac{5}{x}\cdot dx} = e^{-5\log x} = e^{\log x^{-5}} = \frac{1}{x^5}$

$$z(I.F) = \int Q(I.F)\,dx + c$$

$$z\cdot\frac{1}{x^5} = \int -\frac{5x^2}{x^5}dx + c$$

$$\frac{z}{x^5} = -\int\frac{5}{x^3}dx + c$$

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$$= -5\left[\frac{x^{-4+1}}{-4+1}\right] + c$$

$$= -5\left[\frac{x^{-3}}{-3}\right] + c$$

$$\frac{1}{x^5 y^5} = \frac{5}{3x^3} + c$$

Applications of first order D.E.

1) Orthogonal trajectories.
2) Newton's law of cooling.
3) Law of natural growth & decay.

Newton's Law of cooling:

By the Newton's Law of cooling rate of change of temperature of a body is directly proportional to difference of temperature of body and that of surrounding medium.

→ If θ is temperature of body and θ_0 is temperature of surrounding medium then by Newton's Law of cooling.

$$\frac{d\theta}{dt} \propto \theta - \theta_0$$

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\int \frac{d\theta}{\theta - \theta_0} = -\int k\,dt$$

c = constant integration
k = constant of proportionality

$$\log(\theta - \theta_0) = -kt + \log c$$

$$\theta - \theta_0 = ce^{-kt}$$

Q) If temperature of body is changing 100°C to 70°C in 15min find when the temperature will be 40 if the temperature of air is 30°C.

Sol: $\theta_0 = 30°$

$\theta = 100°$, $t = 0$

$\theta = 70°$, $t = 15$

$\theta = 40$, $t = ?$

$(\theta - \theta_0) = ce^{-kt}$ —— (1)

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at $t=0, \theta=100$

$(100-30) = Ce^{-k(0)}$

$70 = C$

at $t=15, \theta=70$

$(70-30) = 70 \cdot e^{-15k}$

$40 = 70 \cdot e^{-15k}$

$\frac{4\cancel{0}}{7\cancel{0}} = e^{-15k}$

$-15k = \log \frac{4}{7}$

$k = -\frac{1}{15} \log \frac{4}{7}$

at $\theta = 40, t = ?$

① $\Rightarrow (40-30) = 70 e^{-kt}$

$\frac{10}{70} = e^{-kt}$

$e^{-kt} = \frac{1}{7}$

$-kt = \log \frac{1}{7}$

$t = -\frac{1}{k} \log \frac{1}{7}$

$= 15 \frac{\log \frac{1}{7}}{\log \frac{4}{7}}$

$t = 52$ min

Law of Natural Growth and Decay:

Let $x(t)$ be the amount of substance present at any time t then rate of change of substance is directly proportional to the amount of substance present at any time 't'

(ie.) $\frac{dx}{dt} \propto x$

$\frac{dx}{dt} = kx$

$\int \frac{dx}{x} = \int k\,dt$

$\log x = kt + c$

$x = ce^{kt}$ (growth)

& $x = ce^{-kt}$ (decay)

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Q) A bacteria culture grows exponentially increases from 200gm to 500gm in the period from 6am to 9am how many grams will be present at noon.

Sol. $x=200, t=0$

$x=500, t=3hr$

$x=?, t=6hr$

By the law of natural growth

$x = ce^{kt}$

$t=0, x=200;$

$\boxed{200=c}$

at $t=3, x=500;$

$50 = 200e^{3k}$

$\log \frac{5}{2} = 3k$

$k = \frac{1}{3}\log\frac{5}{2}$

at $t=6hr, x=?$

$x = 200e^{6k}$

$x = 200e^{\cancel{6}^{2} \cdot \frac{1}{\cancel{3}}\log\frac{5}{2}}$

$x = 200\left(\frac{5}{2}\right)^2$

$= \cancel{200}^{50} \times \frac{25}{\cancel{4}}$

$x = 1250gm.$

Orthogonal Trajectories:

The two curves are said to be orthogonal to each other if angle b/w tangents of corresponding curve is $90°$.

Procedure:

→ Let $y=f(x)$ be the given curve. Find the D.E of given curve.

→ Eliminate the arbitary constant from two curves.

→ Now, Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ which is DE of reqd orthogonal trajectory which on solving gives the eqn of orthogonal trajectory.

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Q) Find the orthogonal trajectory of family of parabolas passing through origin and foci on y-axis.

Sol. The eq^n of family of parabolas is $x^2 = 4ay \rightarrow ①$ where a is the parameter.

D. ① w.r.t. x

$$2x = 4a \frac{dy}{dx}$$

$$a = \frac{x^2}{4y} \text{ from } ①$$

eliminating a from ① & ② we get

$$2x = \frac{x^2}{y} \frac{dy}{dx} \rightarrow ③$$

which is D.E of given curve replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ gives the D.E of orthogonal trajectory.

$$\Rightarrow 2x = -\frac{x^2}{y} \frac{dx}{dy}$$

$$\int 2y\, dy = -\int x\, dx$$

$$\cancel{2} \cdot \frac{y^2}{\cancel{2}} = -\frac{x^2}{2}$$

$$y^2 + \frac{x^2}{2} = c$$

which is the eq^n of req^d orthogonal trajectory.

Q) Find O.T of family of circles passing through origin and centre on x-axis.

Sol. The eq^n of circle passing through origin centre on x-axis is

$$x^2 + y^2 + 2gx = 0 \rightarrow ①$$

$$2x + 2y \frac{dy}{dx} + 2g = 0 \rightarrow ②$$

$$x + y \frac{dy}{dx} + g = 0$$

$$-\left(x + y \frac{dy}{dx}\right) = g$$

$$x^2 + y^2 + 2\left[-\left(x + y\frac{dy}{dx}\right)\right] x = 0$$

$\therefore x^2+y^2-2x^2-2xy\frac{dy}{dx}=0 \rightarrow (3)$

eq (3) is D.E of given curve.

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x^2+y^2-2x^2+2xy\frac{dx}{dy}=0$$

$$-x^2+y^2+2xy\frac{dx}{dy}=0$$

$$(y^2-x^2)dy+2xy\,dx=0 \rightarrow (4)$$

$M=2xy \qquad N=y^2-x^2$

$\frac{\partial M}{\partial y}=2x \qquad \frac{\partial N}{\partial x}=-2x$

$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ is not exact homogeneous.

$$Mx+Ny=2x^2y+y^3-x^2y$$
$$=x^2y+y^3$$

$$\frac{1}{Mx+Ny}=\frac{1}{y(x^2+y^2)}$$

$$\frac{y^2-x^2}{y(x^2+y^2)}dy+\frac{2x\cancel{y}}{\cancel{y}(x^2+y^2)}dx=0$$

$$\frac{y^2-x^2}{y(x^2+y^2)}dy+\frac{2x}{x^2+y^2}dx=0 \rightarrow (5)$$

eq (5) is exact DE the sol.^{n} is given by

$\int M\,dx + \int(\text{terms independent of } x \text{ in } N)\,dy=c$
(y const)

$$\int\frac{2x}{x^2+y^2}dx+0=c$$

$$\log|x^2+y^2|=c$$

$$x^2+y^2=c$$

Q) S.T the system of Parabolas $y^2 = 4a(x+a)$ is self ~~orgo~~ orthogonal.

Sol. $y^2 = 4a(x+a) \rightarrow (1)$

D. w.r.t. x

$2y\frac{dy}{dx} = 4a \cdot 1 \rightarrow (2)$

$a = \frac{y}{2} \cdot \frac{dy}{dx}$

$y^2 = \frac{2}{4}\left(\frac{y}{x} \cdot \frac{dy}{dx}\right)\left(x + \frac{y}{2} \cdot \frac{dy}{dx}\right)$

$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2} \cdot \frac{dy}{dx}\right)$

taking $\frac{dy}{dx} = p$

$y^2 = 2yp\left(x + \frac{yp}{2}\right) \rightarrow (3)$

eq^n 3 is D.E of given curve and on replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ gives the D.E of orthogonal trajectory.

$y^2 = -\frac{2y}{p}\left[x - \frac{y}{2p}\right]$

$y^2 p = -2yx + \frac{2y^2}{2p}$

$y^2 p^2 = -2ypx + y^2$

$y^2 = 2xyp + y^2 p^2$

$= 2yp\left(x + \frac{yp}{2}\right) \rightarrow (4)$

∴ As eqn (3) and (4) are same.

Given D.E is self - orthogonal.

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UNIT - 2

Ordinary Differential Equations of second and Higher Order.

* Homogeneous D.E – $f(D)y = 0$
* Non-Homogeneous D.E – $f(D)y = Q(x)$

where (a) $Q(x) = e^{ax}$

(b) $Q(x) = \cos bx$ (or) $\sin bx$

(c) $Q(x) =$ Polynomial in x

(d) $Q(x) = e^{ax} \cdot V(x)$, where $V(x)$ is x^m (or) $\sin bx$ (or) $\cos bx$

(e) $Q(x) = x \cdot V(x)$ where $V(x)$ is $\cos bx$ (or) $\sin bx$.

* Method of variation of Parameters.
* Equations reducable to constant co-effecients.

a) Legenders equation.

b) cauchys – euler's equation.

Homogeneous Linear D.E : An eq^n of the form

$$\frac{d^n y}{dx^n} + P_1 \cdot \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n \cdot y = Q(x) \quad — ①$$

is known as Linear D.E with constant co-efficient.

→ If $Q(x)=0$ then eq ① is known as linear homogeneous D.E otherwise it is known as non-homogeneous linear D.E of n^{th} order.

Notation:

Taking $\frac{d}{dx} = D$ known as differential operator eq ① can be written as

$$① \Rightarrow D^n y + P_1 D^{n-1} y + P_2 D^{n-2} + \cdots + P_n y = Q(x)$$

$$[D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n] y = Q(x)$$

$$f(D) y = Q(x) \rightarrow ②$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n$

solution of Homogeneous D.E

If $Q(x)=0$ then $f(D)y=0$ is known as homogeneous linear D.E whose sol^n is given as follows.

STEP-1 :- cosider auxilary eq^n $f(m)=0$ which is an n^{th} degree polynomial 'n' will have 'n' roots. Depending on the nature of these 'n' roots we will have 'n' solutions.

consider 3^{rd} order D.E

$[D^3 + P_1 D^2 + P_2 D + D_3] y = 0$ whose auxilary equation is

$f(m) = m^3 + P_1 m^2 + P_2 m + P_3 = 0$ which will have 3 roots α, β, γ.

case i: if 3 roots are real & distinct then sol^n is

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x}.$$

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case ii:- If the 3 roots are real and equal ie, $\alpha=\beta=\gamma$ then solution is $y=[c_1+c_2x+c_3x^2]e^{\alpha x}$.

case iii:- If two roots are repeated and one is distinct then $y=[c_1+c_2x]e^{\alpha x}+c_3e^{\gamma x}=0$ (ie, $\alpha=\beta$ & γ) is distint).

case iv:- If two roots are complex conjugate and one is real (ie, $\alpha=a+ib$, $\beta=a-ib$, $\gamma=$ real) then

$$y=e^{ax}[c_1\cos bx+c_2\sin bx]+c_3e^{\gamma x}.$$

NOTE! If the given D.E of 4^{th} order then we will have 4 roots suppose that four roots are equal complex conjugate roots. (ie, $\alpha=\beta=a+ib$, & $\gamma=\Delta=a-ib$ then

$$y=e^{ax}[(c_1+c_2x)\cos bx+(c_3+c_4x)\sin bx]$$

Q) solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

Sol Taking $\frac{dy}{dx}=0$ the given eq^n can be written as

$$D^2y+Dy+y=0$$

$$y[D^2+D+1]=0$$

Auxilary equation is $m^2+m+1=0$

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$$

$$=\frac{-1\pm\sqrt{3}i}{2}=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$$

$$y=e^{-1/2x}\left[c_1\cos\frac{\sqrt{3}}{2}+c_2\sin\frac{\sqrt{3}}{2}x\right]$$

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Q) $[D^3-14D+8]y=0$

sol A.E. $m^3-14D+8=0$

$m=-4,\ 3.41,\ 0.58$

$$\begin{array}{r|rrrr} -4 & 1 & 0 & -14 & 8 \\ & 0 & -4 & 16 & -8 \\ \hline & 1 & -4 & 2 & \boxed{0} \end{array}$$

$(m+4)(m^2-4m+2)=0$

$$m=\frac{4\pm\sqrt{16-8}}{2}$$

$$=\frac{4\pm\sqrt{8}}{2}=\frac{4\pm2\sqrt{2}}{2}$$

$$=2\pm\sqrt{2}$$

$$y=c_1e^{-4x}+c_2e^{(2+\sqrt{2})x}+c_3e^{(2-\sqrt{2})x}$$

Inverse Operator:

If $D=\frac{d}{dx}$ is the differential operator then $\frac{1}{D}=D^{-1}=\int$ is known as inverse operator.

Note: If Q is a function of x and α is constant then particular value of

$$\frac{1}{D-\alpha}Q=e^{\alpha x}\int Q\cdot e^{-\alpha x}\cdot dx \quad \text{and}$$

$$\frac{1}{D+\alpha}Q=e^{\alpha x}\int Q\,e^{\alpha t}\,.dx$$

Particular Integral:

If $f(D)y=Q(x)$ such that $Q(x)\neq0$ then the particular integral is given by

$$y_p=\frac{1}{f(D)}Q(x).$$

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Solution of non-Homogeneous linear D.E:

If $f(D) \cdot y = Q(x)$ such that $Q(x) \neq 0$ is known as non-Homogeneous linear D.E whose general solution is given by $y = y_c + y_p = C.F + P.I$

→ where y_c is known as complementary function which is a solution of corresponding homogeneous D.E.

→ y_p is known as particular integral which depends on $Q(x)$.

Case i:- If $Q(x) = e^{ax}$ then particular integral

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \Rightarrow f(a) \neq 0.$$

→ If $f(a) = 0 \Rightarrow a$ is a root repeated k times for $k = 1, 2, 3 \ldots$

then $y_p = \frac{x^k}{k!} \cdot \frac{1}{\phi(a)} e^{ax}$

Q) $[D^2+16]y = e^{-4x}$

Sol. A.E is $m^2+16=0$

$m^2 = -16$

$m = \pm 4i$

$y_c = [c_1 \cos 4x + c_2 \sin 4x]$

$$y_p = \frac{1}{D^2+16} e^{-4x}$$

$$= \frac{1}{(-4)^2+16} e^{-4x}$$

$$= \frac{1}{32} e^{-4x}$$

$y = y_c + y_p$

$= [c_1 \cos 4x + c_2 \sin 4x] + \frac{1}{32} e^{-4x}.$

Q) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol) A.E $(m+2)(m-1)^2 = 0$

$m=-2,\ m=1,1$

$y_c = c_1 e^{-2x} + (c_2+c_3x)e^{x}$

$y_p = \frac{1}{(D+2)(D-1)^2} e^{-2x} + \cancel{2}\frac{1}{(D+2)(D-1)^2}\left[\frac{e^x - e^{-x}}{\cancel{2}}\right]$

$= \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{1}{(D+2)(D-1)^2} e^{x} - \frac{1}{(D+2)(D-1)^2} e^{-x}$

$= \frac{1}{(-2-1)^2}\cdot\frac{1}{D+2} e^{-2x} + \frac{1}{1+2}\cdot\frac{1}{(D-1)^2} e^{x} - \frac{1}{(-1+2)(-1-1)^2} e^{-x}$

$= \frac{1}{9}\cdot\frac{x}{1!} e^{-2x} + \frac{1}{3}\cdot\frac{x^2}{2!} e^{x} - \frac{1}{4} e^{-x}.$

Case-2: If $Q(x)$ $\sin bx$ or $\cos bx$.

$y_p = \frac{1}{f(D)} \cos bx$ or $\sin bx$

$y_p = \frac{1}{\phi(D^2)} \cos bx$ or $\sin bx$

put $D^2 = -b^2$

such that $\phi(-b^2) \neq 0$

→ If $\phi(-b^2) = 0$, then $y_p = \frac{1}{\phi(D^2)} \cos bx = \frac{x}{2b}\sin bx$

or $y_p = \frac{1}{\phi(D^2)} \sin bx = -\frac{x}{2b}\cos bx.$

Q) $[D^2-4D+3]y = \sin 3x\cdot\cos 2x.$

Sol. A.E is $m^2-4m+3=0$

$m^2-3m-m+3=0$

$m(m-3)-1(m-3)=0$

$(m-1)(m-3)=0$

$m=1,3$

$y_c = c_1e^{x} + c_2e^{3x}$

$y_p = \frac{1}{D^2-4D+3}\sin 3x\cdot\cos 2x$

$= \frac{1}{D^2-4D+3}\cdot\frac{1}{2}[\sin 5x + \sin x]$

$[\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]]$

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$= \frac{1}{2D^2-4D+3}\sin 5x + \frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin x$

$y_{P_1} = \frac{1}{2}\cdot\frac{1}{D^2-4D+3}\sin 5x$

Put $D^2 = -5^2 = -25$

$= \frac{1}{2}\cdot\frac{1}{-25-4D+3}\sin 5x$

$= -\frac{1}{4}\frac{1}{2D+1}\sin 5x$

$= -\frac{1}{4}\frac{1}{11+2D}\cdot\frac{11-2D}{11-2D}\sin 5x$

$= -\frac{1}{4}\frac{11\sin 5x - 2D\sin 5x}{121-4D^2}$

$= -\frac{1}{4}\frac{11\sin 5x - 2D\sin 5x}{22}$

$= \frac{1}{884}[10\cos 5x - 11\sin 5x]$

$y_{P_2} = \frac{1}{2}\frac{1}{D^2-4D+3}\sin x \Rightarrow D^2 = -1^2 = -1$

$= \frac{1}{2}\frac{1}{-1-4D+3}\sin x$

$= \frac{1}{2}\frac{1}{-4D+2}\sin x$

$= \frac{1}{4}\frac{1}{1-2D}\sin x$

$= \frac{1}{4}\frac{1+2D}{1-4D^2}\sin x$

$= \frac{1}{4}\frac{1+2D}{1+4}\sin x$

$= \frac{1}{4}\frac{\sin x + 2D\sin x}{5}$

$= \frac{1}{20}[\sin x + 2\cos x]$

case iii:-

If $Q(x)$ is a polynomial in x then particular integral

$y_P = \frac{1}{f(D)}Q(x)$.

write $f(D)$ in the form $[1 \pm \phi(D)]$ by taking least degree term common then $\frac{1}{f(D)} = [1 \pm \phi(D)]^{-1}$

Expand using binomial expansion and operate with the terms of expansion.

Expansions.

1) $(1-D)^{-1} = 1+D+D^2+D^3+\cdots$
2) $(1+D)^{-1} = 1-D+D^2-D^3+\cdots$
3) $(1-D)^{-2} = 1+2D+3D^2+4D^3+\cdots$
4) $(1+D)^{-2} = 1-2D+3D^2-4D^3+\cdots$
5) $(1-D)^{-3} = 1+3D+6D^2+10D^3+\cdots$
6) $(1+D)^{-3} = 1-3D+6D^2-10D^3+\cdots$

Q) solve $[D^2+3D+2]y = 2\cos(2x+3) + 2e^x + x^2$

Sol. A.E $\quad m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+(m+2)=0$

$m=-1,-2$

$y_c = c_1e^{-x} + c_2e^{-2x}$

$y_P = \dfrac{1}{D^2+3D+2}\, 2\cos(2x+3) + 2e^x + x^2$

$= \dfrac{1}{D^2+3D+2}\, 2\cos(2x+3) + \dfrac{1}{D^2+3D+2}\, 2e^x + \dfrac{1}{D^2+3D+2}\, x^2$

$y_{P_1} = \dfrac{1}{D^2+3D+2}\, 2\cos(2x+3)$

Put $D^2 = -b^2 = -2^2 = -4$

$= \dfrac{(2x+3)}{-4+3D+2} = \dfrac{1}{3D-2}\, 2\cos(2x+3)$

$= \dfrac{3D+2}{(3D)^2-4}\, 2\cos(2x+3)$

$= \dfrac{6D\cos(2x+3) + 4\cos(2x+3)}{9(-4)-4}$

$= \dfrac{12\sin(2x+3) + 4\cos(2x+3)}{-36-4} = \dfrac{1}{-40}\, 12\sin(2x+3) + 4\cos(2x+3)$

$= \dfrac{1}{-20}\, 3\sin(2x+3) + 2\cos(2x+3)$

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$$y_{P_2} = \frac{1}{D^2+3D+2} 2e^x \quad (\text{or}) \quad \frac{1}{(D+1)(D+2)} 2e^x$$

$$= \frac{1}{1+3+2} 2e^x \Rightarrow \frac{1}{6} 2e^x$$

$$= \frac{1}{3} e^x$$

$$y_{P_3} = \frac{1}{D^2+3D+2} x^2$$

$$= \frac{1}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]} x^2$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2$$

$$[1+D]^{-1} = 1-D+D^2-D^3+\cdots$$

$$\left[\begin{array}{l} Dx^2 = 2x \\ D^2x^2 = 2 \\ D^3x^2 = 0 \end{array}\right]$$

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right] x^2$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{9D^2}{4}+\frac{6D^3}{4}\right] x^2$$

$$= \frac{1}{2}\left[x^2-\frac{2}{2}-\frac{3(2x)}{2}+0+\frac{9(2)}{\cancel{4}_2}\right]$$

$$= \frac{1}{2}\left[x^2-1-3x+\frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

$$y = y_{P_1}+y_{P_2}+y_{P_3}$$

Case iv:

When $Q(x) = e^{ax}.V(x)$ where $V(x)$ is $\cos bx$ (or) $\sin bx$ (or) Polynomial in x then

$$y_P = \frac{1}{f(D)} e^{ax}.V(x) = e^{ax}.\frac{1}{f(D+a)} V(x)$$

Q) solve $[D^4-1]y = e^x.\cos x$

Sol. A.E is $m^4-1=0$

$$(m^2)^2-1^2=0$$

$$(m^2+1)(m^2-1)=0$$

$$m^2=-1 \quad m^2=1$$

$$m=\pm i \quad m=\pm 1$$

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$y_c = c_1\cos x + c_2\sin x + c_3e^{x} + c_4e^{x}$

$y_p = \frac{1}{D^4-1}e^x \cdot \cos x = \frac{1}{(D^2+1)(D^2-1)}e^x\cos x$

Put $D = D+a = D+1$

$= e^x\frac{1}{[(D+1)^2+1][(D+1)^2-1]}\cos x$

$= e^x\frac{1}{[D^2+2D+2][D^2+2D]}\cos x$

Put $D^2 = -b^2 = -1$

$= e^x\frac{1}{[-1+2D+2][-1+2D]}\cos x$

$= e^x\frac{1}{[2D+1][2D-1]}\cos x = e^x\frac{1}{(2D)^2-1^2}\cos x$

$= e^x\frac{1}{4(-1)(-1)}\cos x = \frac{e^x}{-5}\cos x$

Case V :-

If $Q(x) = x \cdot V(x)$ where $V(x)$ is $\sin vx$ or $\cos vx$ then

$y_p = \frac{1}{f(D)}x \cdot V(x) = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V(x).$

Q) solve $[D^2+2D+1]y = x\cos x$

Sol.
$m^2+2m+1=0$

$m^2+m+m+1=0$

$m(m+1)+(m+1)=0$

$(m+1)(m+1)=0$

$m=-1,-1$

$y_c = (c_1+c_2x)e^{-x}$

$y_p = \frac{1}{D^2+2D+1}x\cos x$

$= \left[x - \frac{2D+2}{D^2+2D+1}\right]\frac{1}{D^2+2D+1}\cos x$

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$$= \left[x - \frac{2D+2}{D^2+2D+1}\right] \frac{1}{-1+2D+1} \cos x$$

$$= \frac{1}{2}\left[x - \frac{2D+2}{D^2+2D+1}\right] \sin x$$

$$= \frac{1}{2}\left[x \sin x - \frac{(2D\sin x + 2\sin x)}{-1+2D+1}\right]$$

$$= \frac{1}{2}\left(x \sin x - \frac{1}{2D}(2\cos x + 2\sin x)\right]$$

$$= \frac{1}{2}\left[x \sin x - \int(\cos x + \sin x)\,dx\right]$$

$$= \frac{1}{2}\left[x \sin x - \sin x + \cos x\right]$$

Q) $(D^2-4)y = x \sin \lambda x$

sol A.E eq^n $m^2-4=0$

$m^2-2^2=0$

$m = \pm 2$

$$y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = \frac{1}{D^2-4} x \sin \lambda x$$

$$= \left[x - \frac{2D}{D^2-4}\right] \frac{1}{D^2-4} \sin \lambda x$$

$$= \left[x - \frac{2D}{D^2-4}\right] \frac{1}{-\lambda^2-4} \sin \lambda x$$

$$= -\frac{1}{\lambda^2+4}\left[x \sin \lambda x - \frac{2D}{D^2-4} \sin \lambda x\right]$$

$$= -\frac{1}{\lambda^2+4}\left[x \sin \lambda x - \frac{2\lambda \cos \lambda x}{-\lambda^2-4}\right]$$

$$= -\frac{1}{\lambda^2+4}\left[x \sin \lambda x + \frac{2\lambda \cos \lambda x}{\lambda^2+4}\right]$$

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Method of Variation of Parameter:

If $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R$ is a second ODE which can be solved by method of variation parameters as follows:

STEP-1: Let the complementary function be

$$y_c = c_1 u(x) + c_2 v(x).$$

STEP-2: Let the particular integral $y_p = Au(x) + Bv(x)$.

where, $A = -\int \frac{VR}{\omega} dx$

$$B = \int \frac{VR}{\omega} dx$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

known as wronskian matrix $= \omega$

Q) solve by the method of variation of Parameters $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

sol. A·E $\Rightarrow m^2+1=0$

$$m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

Let $y_p = A\cos x + B\sin x$

Here, $u(x) = \cos x$; $v(x) = \sin x$

$R = \operatorname{cosec} x$; $\omega = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$$A = -\int \frac{VR}{W} dx$$

$$A = -\int \frac{\sin x \cdot \operatorname{cosec} x}{1} dx = -\int \sin x \cdot \frac{1}{\sin x} \cdot dx$$

$$A = -\int 1 \cdot dx = -x$$

$$\boxed{A = -x}$$

$$B = \int \frac{UR}{\omega} dx$$

$$B = \int \cos x \cdot \operatorname{cosec} x \, dx = \int \cos x \cdot \frac{1}{\sin x} dx$$

$$= \int \frac{\cos x}{\sin x} \cdot dx = \int \cot x = \log(\sin x$$

$\boxed{B = \log(\sin x)}$ $\Rightarrow y_p = -x\cos x + \log(\sin x)\sin x \Rightarrow y = y_c + y_p$

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Q) $(D^2-2D+2)y = e^x \tan x$

Sol. $A\cdot E = m^2-2m+2$

$m = 1 \pm i$

$y_c = e^x(c_1 \cos x + c_2 \sin x)$

Let $y_p = Ae^x \cos x + Be^x \sin x$

$u = e^x \cos x,\ V = e^x \sin x,\ R = e^x \tan x$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ -e^x \sin x + \cos x\, e^x & e^x \cos x + \sin x\, e^x \end{vmatrix} = e^{2x} \quad \left[\text{Let} = e^{2x}(\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos)\right]$$

$$A = -\int \frac{e^x \sin x \times e^x \tan x}{e^{2x}}\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x} \cdot dx = -\int \frac{1-\cos^2 x}{\cos x}\, dx$$

$$= -\int (\sec x - \cos x)\, dx = \log|\sec x + \tan x| \mp \sin x$$

$$B = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} \cdot dx$$

$$= \int \sin x\, dx = -\cos x$$

$$y_p = [\log(\sec x + \tan x) - \sin x]\, e^x \cos x - (\cos x) e^x \sin x$$

$$y = y_p + y_c.$$

Euler's cauchy equation:

An equation of the form $(x^n D^n + P_1 x^{n-1} D^{n-1} + \cdots + P_n)\, y = Q(x)$ $P_1, P_2, \ldots P_n$ are constants is known as Euler's cauchy's equation which can be reduced to linear equation with constant co-efficient by substituting $x = e^z$.

Let $x = e^z$

then $z = \log x$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\therefore Dy = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{x} \frac{dy}{dz} \Rightarrow (1)$$

since $D = \frac{d}{dx}$ $||^{ly}$ take $\frac{d}{dz} = \theta$

$① \Rightarrow Dy = \frac{1}{x}\theta y$

$\Rightarrow \boxed{xDy = \theta y}$

Similarly $x^2D^2y = \theta\cdot(\theta-1)y$

$x^3D^3y = \theta(\theta-1)(\theta-2)y$ and soon.

Q) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$ — ①

Sol taking $\frac{d}{dy} = D$

$x^2D^2y - xDy + y = \log x$

Put $\boxed{x = e^z} \Rightarrow z = \log x$

$xDy = \theta y$

$xDy = \theta y$

$x^2D^2y = \theta(\theta-1)y$

$\theta(\theta-1)y - \theta y + y = \log x$

$\theta = \frac{d}{dz}$

$(\theta^2 - \theta - \theta + 1)y = \log x$

$(\theta^2 - 2\theta + 1)y = z \rightarrow ②$

eq② is D.E with constant co-efficient.

$A.E = m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$y_c = (c_1 + c_2 z)e^z$

$y_p = \frac{1}{\theta^2 - 2\theta + 1} z$

$= \frac{1}{(\theta-1)^2} z$

$= \frac{1}{(1-\theta)^2} z = (1-\theta)^{-2} z$

$= [1 + 2\theta + 3\theta^2] z$

$= z + 2$

$x = e^z$,,

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Q) solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left[x+\frac{1}{x}\right] \to (1)$

Taking $\boxed{\frac{d}{dx} = D}$

$[x^3D^3y + 2x^2D^2y + 2y] = 10\left[x+\frac{1}{x}\right]$ — (1)

Taking $x = e^z$

$\log x = z$

$\left(\frac{1}{x} = \frac{dz}{dx}\right)$ X

we have $xDy = \theta y$

$x^2D^2y = \theta(\theta-1)y$

$x^3D^3y = \theta(\theta-1)(\theta-2)y$

where $\boxed{\theta = \frac{d}{dz}}$

(1) $\Rightarrow \theta(\theta-1)(\theta-2)y + 2\theta(\theta-1)y + 2y = 10\left(x+\frac{1}{x}\right)$

$\theta(\theta-1)(\theta-2)y + 2\theta(\theta-1)y + 2y = 10(e^z+e^{-z})$

$(\theta^2-\theta)(\theta-2)y + (2\theta^2-2\theta)y + 2y = 10[e^z+e^{-z}]$

$(\theta^3-\theta^2+2)y = 10(e^z+e^{-z}) \to (2)$

eq (3) is D.E with const co-efficients.

$\therefore$ A.E $m^3-m^2+2=0 \Rightarrow m=-1,\ m=1\pm i$

$y_c = c_1e^{-z} + e^z(c_2\cos z + c_3\sin z)$

$y_p = \frac{1}{\theta^3-\theta^2+2}\, 10(e^z+e^{-z})$

$= 10\, \frac{1}{\theta^3-\theta^2+2}\, e^z + \frac{1}{\theta^3+\theta^2+2}\, e^{-z}$

$= 10\, \frac{1}{1-1+2}\, e^z + 10\, \frac{1}{(\theta^2-2\theta+2)}\, e^{-z}$

$= \frac{10}{2}\, e^z + 10\, \frac{1}{(\theta+1)(\cancel{\theta}1+2+2)}\, e^{-z}$

$= 5e^z + \frac{\cancel{10}^{\,2}}{8} \cdot \frac{z}{1!}\, e^{-z}$

$= 5e^z + 2z\, e^{-z}$

-1	1	-1	0	2
	0	-1	2	-2
	1	-2	2	0

$(\theta+1)(\theta^2-2\theta+2)$

Legendre's Linear equation:

The eq^n of the form $(a+bx)^n D^n y + (a+bx)^{n-1} P D^{n-1} y + \dots + y P_n = Q(x)$.

which can be reduced to D.E with const. co-efficients.

$a+bx = e^z$

$z = \log(a+bx)$

$(a+bx) D = b\theta y$

$(a+bx)^2 D^2 = b^2 \theta(\theta-1) y$

and so on.....

Q) solve $(2x-1)^3 D^3 y + (2x-1) Dy - 2y = x$

Sol Let $2x-1 = e^z \Rightarrow 2x = e^z + 1$

$$x = \frac{e^z+1}{2}$$

$(2x-1) Dy = b\theta y = 2\theta y$

$(2x-1)^2 D^2 y = b^2 \theta(\theta-1) y = 4\theta(\theta-1) y = (4\theta^2 - 4\theta) y$

$(2x-1)^3 D^3 y = b^3 (\theta(\theta-1)(\theta-2) y)$

$= 8\theta(\theta-1)(\theta-2) y$

$= 8[\theta^3 - 3\theta^2 + 2\theta] y$

$\therefore ① \to 8[\theta^3 - 3\theta^2 + 2\theta] y + 2\theta y - 2y = \frac{e^z+1}{2}$

$$y = \frac{e^z+1}{2}$$

A.E $8m^3 - 24m^2 + 18m - 2 = 0$

$m = 1,\ 1 \pm \frac{\sqrt{3}}{2}$

$y_c = c_1 e^z + c_2 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + c_3 e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$

$y_P = \frac{1}{8\theta^3 - 24\theta^2 + 18\theta - 2} \frac{e^z}{2} + \frac{1}{8\theta^3 - 24\theta^2 + 18\theta - 2} \frac{1}{2} e^{z}$

$= \frac{1}{2} \frac{1}{(\theta-1)(8\theta^2 - 16\theta + 2)} e^z + \frac{1}{-2} \frac{1}{2}$

$= \frac{1}{12} \frac{z}{1!} e^z \ \frac{1}{4}$

1	8	-24	18	-2
	0	8	-16	2
	8	-16	2	0

$(\theta-1) \cdot (8\theta^2 - 16\theta + 2)$

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3. Laplace Transforms

Let f(t) be a given function define for all +ve values of 't' then the laplace transforms of f(t) is given by

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

Laplace Transforms of some elimentry function.

1). $L\{1\} = \frac{1}{s}$

$$L\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= \frac{e^{-st}}{-s}\Big|_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^0\right]$$

$$= -\frac{1}{s}[0-1]$$

$$\boxed{L\{1\} = \frac{1}{s}}$$

2) $L\{t\} = \frac{1}{s^2}$

$$L\{t\} = \int_0^\infty e^{-st} \cdot t\,dt = \left\{t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right)\right\}_0^\infty$$

$$= 0 - \left\{0 - \frac{e^0}{s^2}\right\}$$

$$\boxed{L\{t\} = \frac{1}{s^2}}$$

3) $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$L\{t^n\} = \int_0^\infty \underset{u}{e^{-st}}\ \underset{v}{t^n}\,dt \rightarrow (1) \qquad [\int uv = u\int v - \int(u'\int v)\,dv.$$

$$= t^n\left(\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty n t^{n-1}\left(\frac{e^{-st}}{-s}\right)dt$$

$$= (0-0) - \frac{n}{s}\int_0^\infty e^{-st} \cdot t^{n-1}\,dt$$

$$= \frac{n}{s} L(t^{n-1})$$

$$\frac{n}{s} \frac{(n-1)}{s} \cdot \frac{(n-2)}{s} L(t^{n-3})$$

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$$= \frac{n}{s} \cdot \frac{n-1}{s} \; \frac{n-2}{s} \cdots \frac{1}{s} L(1)$$

$$\frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{1}{s} \cdot \frac{1}{s}$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

4) $L\{e^{at}\} = \frac{1}{s-a}$

$$L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} dt$$

$$= \int_0^\infty e^{-s(s-a)t} \cdot dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^\infty$$

$$= -\frac{1}{s-a}\{0-1\}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

5) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

$$L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \cdot dt$$

$$= \int_0^\infty e^{-st} \frac{e^{at}-e^{-at}}{2} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-a)t} - e^{-(s+a)t} \cdot dt$$

$$= \frac{1}{2}\left\{ \frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)} \right\}_0^\infty$$

$$= \frac{1}{2}\left\{ \frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{s+a} \right\}_0^\infty$$

$$= \frac{1}{2}\left\{0 - \left(\frac{1}{s+a} + \frac{1}{s+a}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{sa}{s^2-a^2}\right\}$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2}$$

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6) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt$

$= \int_0^\infty e^{-st} \cdot \frac{e^{at}+e^{-at}}{2} \cdot dt$

$= \frac{1}{2} \int_0^\infty e^{-(s-a)t} + e^{-(s+a)t} \cdot dt$

$= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$

$= \frac{1}{2} \left\{ 0 - \left[\frac{1}{-(s-a)} + \frac{1}{-(s+a)} \right] \right.$

$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right]$

$= \frac{1}{2} \left\{ \frac{s+\cancel{a}+s-\cancel{a}}{(s-a)(s+a)} \right\}$

$= \frac{1}{2} \left(\frac{2s}{(s-a)(s+a)} \right)$

$= \frac{s}{s^2-a^2}$.

a) $L\{3\cos 3t \cdot \cos 4t\}$

sol $\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$= L\{ \frac{3}{2}(\cos 7t + \cos t)]$

$= \frac{3}{2} \left[\frac{s}{s^2+7^2} + \frac{s}{s^2+1} \right]$

$= \frac{3}{2} \left[\frac{s}{s^2+49} + \frac{s}{s^2+1} \right]$

$= \frac{3}{2} \left\{ \frac{s^3+s+s^3+49s}{(s^2+49)(s^2+1)} \right\}$

$= \frac{3}{2} \left\{ \frac{2s^3+49s+s}{(s^2+49)(s^2+1)} \right\}$

$= \frac{3}{2} \left\{ \frac{2s^3+50s}{(s^2+49)(s^2+1)} \right\}$

Q) $L\{\cos at\} = \frac{s}{s^2+a^2}$

Sol. $L\{\cos at\} = \int_0^\infty e^{-st} \cos at \cdot dt$

$\left[\int e^{ax} \cdot \cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]\right]$

$= \frac{e^{-st}}{s^2+a^2}\left[-s \cdot \sin at + a\cos at\right]_0^\infty$

$= 0 - \frac{1}{s^2+a^2}\left[-s+0\right] = \frac{s}{s^2+a^2}$ //

Q) $L\{\cos^3 2t\}$

Sol.

$\cos 3t = 4\cos^3 t - 3\cos t$

$\cos^3 t = \frac{1}{4}\left[\cos 3t + 3\cos t\right]$

$L\{\cos^3 2t\} = \frac{1}{4}\left[\cos 6t + 3\cos 2t\right]$

$= \frac{1}{4} \cdot \frac{s}{s^2+36} + \frac{3}{4} \cdot \frac{s}{s^2+4}$

$= \frac{1}{4}\left[\frac{s(s^2+4) + 3s(s^2+36)}{(s^2+36)(s^2+4)}\right]$

$= \frac{1}{4}\left(\frac{s^3+4s+3s^3+108s}{(s^2+36)(s^2+4)}\right)$

$= \frac{1}{4}\left(\frac{4s^3+112s}{(s^2+36)(s^2+4)}\right)$

$= \frac{s\left[4s^2+112\right]}{4(s^2+36)(s^2+4)}$ //

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• Linear property: If $L\{f(t)\} = \bar{f}(s)$ and

$L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}$

$\Rightarrow c_1 \bar{f}(s) + c_2 \bar{g}(s)$

Q) $L\{\sinh^3 2t\}$

$\sinh t = \frac{e^t - e^{-t}}{2}$

$\sinh 2t = \frac{e^{2t} - e^{-2t}}{2}$

$\sinh^3 2t = \left[\frac{e^{2t} - e^{-2t}}{2}\right]^3$

Sol. $L\{\sinh^3 2t\}$

$= \frac{1}{8} L\{e^{6t} - e^{-6t} + 3e^{-2t} - 3e^{2t}\}$

$= \frac{1}{8} L\left[e^{6t} - e^{-6t} + 3e^{2t}(e^{-4t}) - 3e^{4t}e^{-2t}\right]$

$= \frac{1}{8}\left[\frac{1}{s-6} - \frac{1}{s+6} + 3\frac{1}{s+2} - \frac{3}{s-2}\right]$

$= \frac{1}{8}\left[\frac{\cancel{s}+6-\cancel{s}+6}{s^2-36} + 3\left[\frac{\cancel{s}-2-\cancel{s}-2}{s^2-2^2}\right]\right]$

$= \frac{1}{8}\left[\frac{12}{s^2-36} + 3\left(\frac{-4}{s^2-4}\right)\right]$

$= \frac{1}{8}\left[\frac{12}{s^2-36} - \frac{12}{s^2-4}\right] = \frac{12}{8}\left[\frac{s^2-4-s^2+36}{(s^2-36)(s^2-4)}\right]$

$= \frac{3}{2}\left[\frac{32}{(s^2-36)(s^2-4)}\right] = \frac{48}{(s^2-36)(s^2-4)}$

Q) Find the laplace transform of $f(t) = \begin{cases} \sin t \ ; & 0 < t < \pi \\ 0 \ ; & t > \pi \end{cases}$

Sol. W.K.T $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\pi e^{-st} \sin t\,dt + \int_\pi^\infty 0.\,dt$

$= \int_0^\pi e^{-st} \sin t\,dt \qquad \left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

$= \frac{e^{-st}}{(-s)^2+(1)^2}\left[-s\sin t - \cos t\right]_0^\pi$

$= \frac{e^{-st}}{s^2+1}\left[-s\sin\pi - \cos\pi\right] - \frac{e^0}{s^2+1}\left[-s\sin 0 - \cos 0\right]$

$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{1}{s^2+1}(e^{-s\pi}+1)$

First shifting:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{e^{at}.f(t)\} = \bar{f}(s-a)$

Q) $L\{e^{-t}\cos 2t\}$

Sol. $L(\cos 2t)|_{s\to s+1}$

$\frac{s}{s^2+2^2}\Big|_{s\to s+1}$

$\frac{s}{s^2+4} = \frac{(s+1)}{(s+1)^2+4} = \frac{s+1}{s^2+2s+1+4} = \frac{s+1}{s^2+2s+5}$

Q) Find $L\{\sqrt{t}\cdot e^{-3t}\}$

$= L\{\sqrt{t}\}|_{s\to s+3}$

$= \frac{\Gamma\frac{1}{2}+1}{s^{1/2+1}}\Big|_{s\to s+3}$

$= \frac{\Gamma\frac{3}{2}}{s^{3/2}}\Big|_{s\to s+3} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{(s+3)^{3/2}}$

$= \frac{1}{2}\frac{\sqrt{\pi}}{(s+3)^{3/2}}$

Second shifting Property:

If $L\{f(t)\} = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a); & t>a \\ 0; & t<a \end{cases}$ then

$L\{g(t)\} = e^{-as}\bar{f}(s)$.

Q) Find the Laplace transform of $g(t) = \begin{cases} \cos(t-\frac{\pi}{3}); & t>\frac{\pi}{3} \\ 0; & t<\frac{\pi}{3} \end{cases}$

Sol. Here $f(t) = \cos t$

$L\{f(t)\} = L(\cos t) = \frac{s}{s^2+1}$

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Given $f(t-a) = \cos\left(t - \frac{\pi}{3}\right)$

$\Rightarrow a = \pi/3$

$\therefore$ By second shifting theorem $L\{g(t)\} = e^{-as}\bar{f}(s)$

$= e^{-\pi/3 s} \cdot \frac{s}{s^2+1}$

Unique step function:

The unique step function is defined as $\begin{cases} u(t-a) = 0 ; & t < a \\ = 1 ; & t > 1 \end{cases}$

then $L\{u(t-a)\} = \frac{e^{-as}}{s}$

Q) Find the L.T of $e^{-3t} \cdot u(t-2)$

sol. $e^{-3t(t-2)-6} \cdot u(t-2)$

$e^{-6} \cdot e^{-3(t-2)} \cdot u(t-2)$

consider $e^{-6} L\{e^{-3(t-2)} \cdot u(t-2)\}$

Here $f(t) = e^{-3t} \Rightarrow L\{f(t)\} = \frac{1}{s+3} = \bar{f}(s)$

and $a = 2$

$\therefore$ By second shifting theorem.

$e^{-6} L\{e^{-3(t-2)} u(t-2)\} = e^{-6} \cdot e^{-as} \cdot \bar{f}(s)$

$= e^{-6} e^{-2s} \cdot \frac{1}{s+3}$

Another form of second shifting:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(t-a)\, u(t-a)\} = e^{-as}\bar{f}(s)$

Q) Find Laplace transform of $(t-2)^3 u(t-2)$

Here $f(t) = t^3$

$L\{f(t)\} = \frac{3!}{s^4} = \bar{f}(s)$

here $a = 2$

$\therefore$ By second shifting

$L\{(t-2)^3 u(t-2)\} = e^{-as} \cdot \bar{f}(s) \Rightarrow e^{-2s} \cdot \frac{3!}{s^4}$.

change of scale property:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$= \frac{1}{a} L\{f(t)\}\Big|_{s \to s-a}$$

Q) If $L\{f(t)\} = \frac{1}{s} e^{-1/s}$ then find $L\{e^{-t} f(3t)\}$

Sol consider $L\{e^{-t} f(3t)\}$

By the first shifting theorem.

$$L\{e^{-t} f(3t)\} = L\{f(3t)\}\Big|_{s \to s+1} \quad -①$$

By the change of scale property.

$$L\{f(3t)\} = \frac{1}{3} L\{f(t)\}\Big|_{s \to \frac{s}{3}}$$

$$= \frac{1}{3}\left[\frac{1}{s} e^{-1/s}\right]_{s \to \frac{1}{3}}$$

$$= \frac{1}{3}\left[\frac{3}{s} e^{-3/s}\right]$$

$$= \frac{1}{s} e^{-3/s}$$

$\therefore$ eq ① $\Rightarrow L\{e^{-t} f(3t)\} = L\{f(3t)\}_{s \to s+1}$

$$= \left[\frac{1}{s} \cdot e^{-3/s}\right]_{s \to s+1}$$

$$= \frac{1}{s+1} \cdot e^{-3/s+1}$$

Laplace Transforms of derivatives:

If $f(t)$ is continuous function then

$L\{f'(t)\} = s\,\bar{f}(s) - f(0)$ where $\bar{f}(s) = L\{f(t)\}$.

$\rightarrow$ In general

$$\boxed{L\{f^{(n)}(t)\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - f^{(n-1)}(0)}$$

Laplace Transforms of Integral:

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)$

$$= \frac{1}{s} L\{f(t)\}$$

Q) Find $L\left\{\int_0^t e^{-t} \cos t \, dt\right\}$

Sol: By L.T of integrals.

$$L\left\{\int_0^t e^{-t} \cos t \, dt\right\} = \frac{1}{s} L\{f(t)\}$$

$$= \frac{1}{s} L\{e^{-t} \cos t\}$$

$$= \frac{1}{s} [L(\cos t)]_{s \Rightarrow s+1} \quad (1^{st} \text{ shifting})$$

$$= \frac{1}{s} \cdot \left[\frac{s}{s^2+1^2}\right]_{s \to s+1}$$

$$= \frac{1}{s} \left[\frac{s+1}{s^2+2s+1+1}\right] = \frac{1}{s} \left(\frac{s+1}{s^2+2s+2}\right)$$

Q) Find $L\left\{\int_0^t \int_0^t \cosh at \, dt \, dt\right\}$

Here $f(t) = \cosh at$

$$\Rightarrow L\{f(t)\} = \frac{s}{s^2-a^2}$$

$$L\left\{\int_0^t \cosh at \, dt\right\} = \frac{1}{s} L(\cosh at)$$

$$= \frac{1}{s}\left(\frac{s}{s^2-a^2}\right) = \frac{1}{s^2-a^2}$$

$$\Rightarrow L\left\{\int_0^t \int_0^t \cosh at \, dt\right\} = \frac{1}{s} L\left\{\int_0^t \cosh at \, dt\right\}$$

$$= \frac{1}{s}\left[\frac{1}{s^2-a^2}\right]$$

NOTE!

→ In general $L\left\{\int_0^t \int_0^t \int_0^t \cdots f(u)\,du\,du \cdots n \text{ times}\right.$

$$= \frac{1}{s^n} \bar{f}(s) = \frac{1}{s^n} L\{f(t)\}$$

Multiplication by t^n:

If $L\{f(t)\} = \bar{f}(s)$ then $L\{t \cdot f(t)\} = \frac{-d}{ds}\bar{f}(s) = -f^{-1}(s)$.

In general $\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)}$

Q) $L\{t^2 e^{-t} \cos 2t\}$

sol. $L\{\cos 2t\} = \frac{s}{s^2+4}$

$$L\{t^2 \cos 2t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)^2(-2s) - (4-s^2)\,(2(s^2+4)(2s))}{(s^2+4)^2}$$

$$= \frac{(s^2+4)^2(-2s) - (4-s^2)\,(4s(s^2+4))}{(s^2+4)}$$

$$= \frac{(s^2+4)(-2s) - (4-s^2)\,4s}{(s^2+4)^3}$$

$$= \frac{-2s^3 - 8s - 16s + 4s^3}{(s^2+4)^3}$$

$$= \frac{2s^3 - 24s}{(s^2+4)^3}$$

$L(t^2 \cos 2t)_{s \to s+1}$

$$\frac{2(s+1)^3 - 24(s+1)}{((s+1)^2+4)^3}$$

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Division by L:

If $L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty \bar{f}(s)\,ds$

In general $L\left\{\frac{f(t)}{t^n}\right\} = n\int_s^\infty\int_s^\infty \bar{f}(s)\,ds\,ds \cdots n \text{ times}$

Q) Find $L\left\{\frac{\sin 3t \cos t}{t}\right\}$

Sol. $L\{\sin 3t \cdot \cos t\} = L\{\sin 4t + \sin 2t\}$

$= \frac{4}{s^2+16} + \frac{2}{s^2+4}$

$L\left\{\frac{\sin 3t \cos t}{t}\right\} = \frac{1}{2}\int_s^\infty L(\sin 3t \cdot \cos t)\,ds$

$= \frac{1}{2}\int_s^\infty \frac{4}{s^2+16} + \frac{2}{s^2+4}\,ds$

$= \frac{1}{2}\left(\frac{4}{4}\tan^{-1}\left(\frac{s}{4}\right) + \frac{2}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \frac{1}{2}\left\{\left[\tan^{-1}(\infty) + \tan^{-1}(\infty)\right] - \left[\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]\right\}$

$= \frac{1}{2}\left\{\left[\frac{\pi}{2} + \frac{\pi}{2}\right] - \left[\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right]\right\}$

$= \frac{1}{2}\left[\pi - \left(\tan^{-1}\left(\frac{s}{4}\right) + \tan^{-1}\left(\frac{s}{2}\right)\right)\right]$

Q) Find $L\left\{e^{-3t}\int_0^t \frac{\sin t}{t}\,dt\right\}$

Sol. $L(\sin t) = \frac{1}{s^2+1}$

$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{s^2+1}\,ds$

$= \left[\tan^{-1}(s)\right]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1} s$

$= \cot^{-1} s$

$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s} L\left\{\frac{\sin t}{t}\right\}$

$= \frac{1}{s}\cot^{-1} s$

$\therefore L\left\{e^{-3t}\int_0^t \frac{\sin t}{t}\,dt\right\} = L\left\{\int_0^t \frac{\sin t}{t}\right\}\Big|_{s \to s+3}$

$= \frac{1}{s+3}\cot^{-1}(s+3)$

Evaluation of integrals by Laplace Transform:

Q) $\int_0^\infty \frac{\sin 2t}{t} \cdot dt$

By definition. $L f(t) = \int_0^\infty e^{-st} f(t)\, dt$

Comparing the given eq^n with definition $f(t) = \frac{\sin 2t}{t}$, $s=0$.

$\therefore \int_0^\infty \frac{\sin 2t}{t}\, dt = L\left\{\frac{\sin 2t}{t}\right\}_{s=0}$ — ①

$L(\sin 2t) = \frac{2}{s^2+4}$

$L\left(\frac{\sin 2t}{t}\right) = \int_s^\infty \frac{2}{s^2+4}$

$= \left(\cancel{2}\tan^{-1}\left(\frac{s}{2}\right)\right)_s^\infty$

$= \left(\tan^{-1}\left(\frac{s}{2}\right)\right)_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$

$= \cot^{-1}\left(\frac{s}{2}\right)$

$\int_0^\infty \frac{\sin 2t}{t}\, dt = L\left\{\frac{\sin 2t}{t}\right\} = 0$

$= \cot^{-1}\left(\frac{s}{2}\right)\Big|_{s=0}$

$= \cot^{-1}\left(\frac{0}{2}\right)\Big|$

$= \cot^{-1}(0)$

$= \frac{\pi}{2}$

Laplace transform of Periodic function:

Periodic function: A function $f(t)$ is said to be periodic of Period T if $f(t+T) = f(t+2T) = f(t+3T) = \cdots = f(t+nT)$

ex: $\sin x$, $\cos x$ are periodic f^n of period $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots = \sin(x+2\pi x)$ in is Integer.

→ $\tan\theta$ and $\cot\theta$ are periodic of Period π.

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Laplace Transform of periodic function of $f(t)$ of Period T is given by.

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} (f(t))\, dt.$$

Q) $f(t) = 1 \;;\; 0 < t < a/2$

$f(t) = -1 \;;\; a/2 < t < a$

Sol $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

Period $= a$

$$= \frac{1}{1-e^{-sa}} \left\{ \int_0^{a/2} e^{-st}(1)\, dt + \int_{a/2}^{a} e^{-st}(-1)\, dt \right\}$$

$$= \frac{1}{1-e^{-sa}} \left\{ \frac{e^{-st}}{-s}\Big|_0^{a/2} + \frac{e^{-st}}{s}\Big|_{a/2}^{a} \right\}$$

$$= \frac{1}{1-e^{-sa}} \left\{ \left(\frac{e^{-as/2}}{-s} + \frac{1}{s} \right) + \left(\frac{e^{-as}}{s} - \frac{e^{-as/2}}{s} \right) \right\}.$$

U.L L.L

$$= \frac{1}{1-e^{-sa}} \left\{ -\frac{e^{-as/2}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-as/2}}{s} \right\}$$

$$= \frac{1}{s(1-e^{-sa})} \left\{ 1 + e^{-as} - 2e^{-as/2} \right\}$$

$$= \frac{(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})}$$

$$= \frac{(1-e^{-as/2})}{s(1+e^{-as/2})}$$

Inverse Laplace Transforms:

→ If $\bar{f}(s)$ is laplace Transform of $f(t)$ then $f(t)$ is known as inverse laplace transform of $\bar{f}(s)$, denoted by $L^{-1}\{\bar{f}(s)\} = f(t)$

Inverse Laplace Transform of elimentry function:

→ $L^{-1}\left(\frac{1}{s}\right) = 1$

→ $L^{-1}\left(\frac{1}{s^2}\right) = t$

→ $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

→ $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

→ $L^{-1}\left(\frac{1}{s-a}\right) = e^{+at}$

→ $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

→ $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

→ $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$

→ $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

→ $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$.

Linear Property:

If $\bar{f}_1(s)$ & $\bar{f}_2(s)$ are inverse laplace transforms of $f_1(t)$ & $f_2(t)$ and $L^{-1}\{c_1\bar{f}_1(s) + c_2\bar{f}_2(s)\} = c_1 L^{-1}\{\bar{f}_1(s)\} + c_2 L^{-1}\{\bar{f}_2(s)\}$.

Q) $L^{-1}\left\{\frac{s^2+2s-4}{(s^2+9)(s-5)}\right\}$

sol Consider $\frac{s^2+2s-4}{(s^2+9)(s-5)} = \frac{As+B}{s^2+9} + \frac{C}{s-5}$

$s^2+2s-4 = As(s-5) + B(s-5) + C(s^2+9)$

$s=0$; $-4 = 5B + 9C \rightarrow$ (1)

$s=5$ $\quad 31 = 34C$

$C = \frac{31}{34}$ sub in eq (1)

coefficient of s^2

$1 = A + C$

$A = 1 - \frac{31}{34}$

$\boxed{A = \frac{3}{34}}$

coefficient of s:

$2 = -5A + B$

$2 = \frac{-15}{34} + B$

$B = 2 + \frac{15}{34}$

$\boxed{B = \frac{83}{34}}$

$$L^{-1}\left\{\frac{s^2+2s-4}{(s^2+9)(s-5)}\right\} = L^{-1}\left\{\frac{3}{34(s^2+9)} + \frac{83}{34(s^2+9)} + \frac{31}{34(s-5)}\right\}$$

First shifting Theorem:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$.

Q) $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol

$$L^{-1}\left\{\frac{s}{(s^2)^2+(2a^2)^2}\right\}$$

$$L^{-1}\left\{\frac{s}{(s^2+2a^2)^2-4a^2s^2}\right\}$$

$$L^{-1}\left\{\frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}\right\}$$

$$\frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$$

$$= \frac{1}{4a}\left[\frac{1}{s^2+2a^2+2as} - \frac{1}{s^2+2a^2-2as}\right]$$

$$= \frac{1}{4a}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{s^2+a^2+a^2-2(a)(s)} - \frac{1}{s^2+a^2+a^2+2(a)(s)}\right\}$$

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - \frac{1}{4a} L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a} e^{at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\} - \frac{1}{4a} e^{-at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= \frac{e^{at}}{4a^2}\sin at - \frac{e^{-at}}{4a}\sin at$$

$$= \frac{1}{4a^2} e^{at}\sin at - \frac{1}{4a^2} e^{-at}\sin at$$

$$= \frac{1}{2a^2} \cdot \sin at \left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$= \frac{1}{2a^2} \sin at \cdot \sinh t$$

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Second shifting theorem:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{e^{-as}\bar{f}(s)\} = G(t)$,

$$G(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

Q) $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Sol By S.S.T

$a = 3, \quad \bar{f}(s) = \frac{1}{(s-4)^2}$

$$f(t) = L^{-1}\left\{\frac{1}{(s-4)^2}\right\}$$

$$= e^{4t} L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= e^{4t} \cdot t = te^{4t}$$

$$f(t-a) = (t-3)\, e^{4(t-3)}$$

$$= (t-3)\, e^{4t-12}$$

$$\therefore L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = \begin{cases} (t-3)\, e^{4(t-3)} & ; t > 3 \\ 0 & ; t < 3 \end{cases}$$

Change of scale property:

If $L\{f(t)\} = \bar{f}(s)$ then $L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$ for $a > 0, t > 0$

Q) If $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ find $L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\}$

Sol $L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = L^{-1}\left\{\frac{e^{-a/s}}{\left(\frac{s}{a}\right)^{1/2} a^{1/2}}\right\}$

$$= L^{-1}\left\{\frac{e^{-1/(\frac{s}{a})}}{a^{1/2}\left(\frac{s}{a}\right)^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{a}} L^{-1}\left\{\frac{e^{-1/(\frac{s}{a})}}{\left(\frac{s}{a}\right)^{1/2}}\right\}$$

$$\boxed{\begin{array}{l} L^{-1}\{\bar{f}(as)\} = \frac{1}{a} f(t/a) \\ L^{-1}\{\bar{f}(\frac{s}{a})\} = af(at) \end{array}}$$

$$L^{-1}\left\{\frac{e^{-a/s}}{a^{1/2}\left(\frac{s}{a}\right)^{1/2}}\right\} = \frac{1}{a^{1/2}} L^{-1}\left\{\frac{e^{-a/s}}{(s/a)^{1/2}}\right\} = \frac{1}{a^{1/2}} \cdot a \frac{\cos 2\sqrt{at}}{\sqrt{\pi a t}}$$

$$= \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

* Inverse Laplace transform of derivatives.

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\bar{f}^{(n)}(s)\} = (-1)^n t^n f(t)$

Q) Find $L^{-1}\{\log(1+\frac{1}{s^2})\}$

Consider $\bar{f}(s) = \log(1+\frac{1}{s^2})$

$= \log\left(\frac{s^2+1}{s^2}\right)$

$= \log(s^2+1) - \log s^2$.

$\bar{f}'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2}$

$= \frac{2s}{s^2+1} - \frac{2}{s}$

taking L^{-1} on b.s

$L^{-1}\{\bar{f}'(s)\} = L^{-1}\{\frac{2s}{s^2+1} - \frac{2}{s}\}$

$= L^{-1}\{\frac{2s}{s^2+1}\} - 2L^{-1}\{\frac{1}{s}\}$

$= 2\cos t - 2$

By I.L.T of D

$(-1)^1 t^1 f(t) = 2\cos t - 2$

$-t\, L^{-1}\{\bar{f}(s)\} = 2\cos t - 2$

$-t\, L^{-1}\{\bar{f}(s)\} = 2\cos t - 2$

$L^{-1}\{\log(1+\frac{1}{s^2})\} = \frac{2-2\cos t}{t}$

Inverse Laplace Transforms of integrals:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\{\int_s^\infty \bar{f}(s)\,ds\} = \frac{f(t)}{t} = \frac{L^{-1}\{\bar{f}(s)\}}{t}$

$t\, L^{-1}\{\int_s^\infty \bar{f}(s)\,ds\} = L^{-1}\{\bar{f}(s)\}$.

Q) Find $L^{-1}\{\frac{2}{(s-a)^3}\}$

Let $\bar{f}(s) = \frac{2}{(s-a)^3}$

By L.T of integral.

$t \cdot L^{-1}\{\int_s^\infty \bar{f}(s)\,ds\} = L^{-1}\{\bar{f}(s)\}$

$L^{-1}\{\frac{2}{(s-a)^3}\} = t\, L^{-1}\{\int_s^\infty \frac{2}{(s-a)^3}\,ds\}$

$$= 2t \left\{ \frac{(s-a)^{-2}}{-2} \right]_s^\infty \Big\}$$

$$\boxed{\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1}}$$

$$= -t \left\{ \frac{1}{(s-a)^2} \right]_s^\infty \Big\}$$

$$= -t L^{-1} \left\{ 0 - \frac{1}{(s-a)^2} \right\}$$

$$= t L^{-1} \left\{ \frac{1}{(s-a)^2} \right\}$$

$$= t e^{at} L^{-1} \left\{ \frac{1}{s^2} \right\}$$

$$= t^2 e^{at}$$

Multiplication by Powers of s:

If $L^{-1}\{\bar{f}(s)\} = f(t)$ and $f(0) = 0$ then $L^{-1}\{s\bar{f}(s)\} = f'(t) = \frac{d}{dt} L^{-1}\{\bar{f}(s)\}$.

→ In general $L^{-1}\{s^n \bar{f}(s)\} = f^{(n)}(t)$ where $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots, n-1$

Q) Find $L^{-1}\left\{ \frac{s}{(s-4)^5} \right\}$

Sol. By the multiplication by s

$$L^{-1}\{s\bar{f}(s)\} = f'(t) = \frac{d}{dt} L^{-1}\{\bar{f}(s)\}$$

$$L^{-1}\left\{ s \cdot \frac{1}{(s-4)^5} \right\} = \frac{d}{dt} L^{-1}\left\{ \frac{1}{(s-4)^5} \right\}$$

$$L^{-1}\left\{ s \cdot \frac{1}{(s-4)^5} \right\} = \frac{d}{dt} \left\{ e^{4t} \frac{t^4}{4!} \right\}$$

$$= \frac{1}{24} \cdot \frac{d}{dt} \cdot e^{4t} \cdot t^4$$

$$= \frac{1}{24} \left[4e^{4t} t^4 + e^{4t} 4t^3 \right]$$

$$= \frac{1}{24} \left[4e^{4t} t^4 + 4t^3 e^{4t} \right]$$

$$= \frac{1}{6} \left[e^{4t} t^4 + t^3 e^{4t} \right]$$

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Division by s :-

- If $L^{-1}\{\bar{f}(s)\} = f(t)$ then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t) \cdot dt$.

→ In general $L^{-1}\left\{\frac{\bar{f}(s)}{s^n}\right\} = \int_0^t \int_0^t \cdots f(t)\, dt\, dt \cdots$ n times.

Q) $L^{-1}\left\{\frac{1}{s^3(s^2+a^2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t L^{-1}\{\bar{f}(s)\} \cdot dt$

$= \int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right) dt$

$= \int_0^t \frac{1}{a} \sin at \, dt$

$= \frac{1}{a} - \frac{\cos at}{a} \Big|_0^t$

$= -\frac{1}{a^2}[\cos at - 1]$

$= -\frac{1}{a^2}\cos at + \frac{1}{a^2}$

$L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\} = \int_0^t \left[\int_0^t L^{-1}\{\bar{f}(s)\}\, dt\right] dt$

$= \frac{1}{a^2}\int_0^t (-\cos at + 1)\, dt$

$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t$

$= \frac{1}{a^2}\left[t - \frac{\sin at}{a} - 0\right]$

$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]$

$L^{-1}\left\{\frac{1}{s^3(s^2+a^2)}\right\} = \int_0^t \left[\int_0^t \int_0^t L^{-1}\bar{f}(s)\, dt\, dt\, dt\right]$

$= \int_0^t \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]$

$= \frac{1}{a^2}\left[\frac{t^2}{2} + \frac{1}{a}\frac{\cos at}{a}\right]$

$= \frac{1}{a^2}\left[\frac{t^2}{2} + \frac{\cos at}{a^2}\right]_0^t$

$= \frac{1}{a^2}\left[\frac{t^2}{2} + \frac{\cos at}{a^2} - \frac{1}{a^2}\right]$

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Convolution Theorem:

If $L\{f(t)\} = \bar{f}(s)$ & $L\{g(t)\} = \bar{g}(s)$ then $L\{f(t) * g(t)\} = \bar{f}(s) \cdot \bar{g}(s)$.

$\Rightarrow L^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = f(t) * g(t)$

where $f(t) * g(t) = \int_0^t f(u) \cdot g(t-u) du$

a) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol. $L^{-1}\left\{\frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right\}$

$\bar{f}(s) = \frac{s}{s^2+a^2}$

$f(t) = L^{-1}\left\{\frac{s}{s^2+a^2}\right\}$

$= \cos at$

By convolution theorem.

$L^{-1}\{\bar{f}(s) \cdot \bar{g}(s)\} = \int_0^t f(u) \cdot g(t-u) du$

$= \frac{1}{a}\int_0^t \cos au \cdot \sin a(t-u) du$

$= \frac{1}{2a}\int_0^t \sin(\cancel{au} + at - \cancel{au}) - \sin(au - at + au) du$

$= \frac{1}{2a}\int_0^t \sin at - \sin(2au - at) du$

$= \frac{1}{2a}\sin at \int_0^t du - \frac{1}{2a}\int_0^t \sin(2au - at) du$

$= \frac{1}{2a}\sin at [u]_0^t + \frac{1}{2a}\left[\frac{\cos(2au-at)}{2a}\right]_0^t$

$= \frac{1}{2a}\sin at [t] + \frac{1}{4a^2}\left[\cos(2au-at)\right]_0^t$

$= \frac{t \sin at}{2a} + \frac{\cos(2at-at) - \cos(2(0)-at)}{4a^2}$

$= \frac{t \sin at}{2a} + \frac{1}{4a^2}\cos at - \cos at = \frac{t \sin at}{2a}$

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Applications of Linear to Differential equations:

→ Laplace transforms of derivatives.

$$\boxed{L\{f^{(n)}t\} = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots f^{(n-1)} f(0)\}}$$

Q) $y'' = t\cos 2t$ with $y(0) = 0$ and $y'(0) = 0$

Sol Taking Laplace on b.s $L(y'') = L(t\cos 2t)$

$$[s^2 L(y) - 3y(0) - y'(0)] = \frac{-d}{ds} L(\cos 2t)$$

Given $y(0) = 0$ & $y'(0) = 0$

$$s^2 \cdot L(y) = \frac{-d}{ds}\left\{\frac{s}{s^2+4}\right\}$$

$$= \left\{\frac{(s^2+4)\cdot 1 - s(2s)}{(s^2+4)^2}\right\}$$

$$= \left\{\frac{4-s^2}{(s^2+4)^2}\right\} = \frac{s^2-4}{(s^2+4)^2}$$

$$L(y) = \frac{s^2-4}{s^2(s^2+4)}$$

Consider $\dfrac{s^2-4}{s^2(s^2+4)}$

$$\frac{s^2-4}{s^2(s^2-4)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{(s^2+4)^2}$$

$$s^2 - 4 = As(s^2+4)^2 + B(s^2+4)^2 + Cs^3(s^2+4) + Ds^2(s^2+4) + Es^3 + Fs^2.$$

$$s^2 - 4 = As[s^4+16+8s^2] + B[s^4+16+8s^2] + C[s^2+4s^3] + D[s^2+4s^2] + Es^3 + Fs^2.$$

Put $s = 0$;

$$-4 = 16B$$

$$\boxed{B = -\frac{1}{4}}$$

Coefficient of s: $\boxed{A = 0}$

coefficient of s^2: $8B + 4D + F = 1$

$4D + F = 3 \longrightarrow (1)$

Coefficient of s^3: $A + 4C + E = 0$

$4C + E = 0 \longrightarrow (2)$

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coefficient of s^4 : $B+D=0$

$\boxed{D=\frac{1}{4}}$

coefficient of s^5 :

$0=A+C$

$\boxed{C=0}$

(2) eq $\Rightarrow E=0$

(1) eq $\Rightarrow 4\left(\frac{1}{4}\right)+E=3$

$\boxed{E=2}$

$L(y)=\frac{-1}{4s^2}+\frac{1}{4(s^2+4)}+\frac{2}{(s^2+4)^2}$

$y=L^{-1}\left\{\frac{-1}{4s^2}\right\}+L^{-1}\left(\frac{1}{4(s^2+4)}\right)+L^{-1}\left(\frac{1}{(s^2+4)^2}\right)$

$$\boxed{\frac{1}{(s^2+a^2)^2}=\frac{1}{2a^3}(\sin at-at\cos at)}$$

$y=\frac{-t}{4}+\frac{1}{8}\sin 2t+\frac{2}{2(2)^3}\left[\sin 2t-2t\cos 2t\right]$

$y=\frac{-t}{4}+\frac{1}{8}\sin 2t+\frac{1}{8}\sin 2t-\frac{t}{4}\cos 2t$

$y=\frac{-t}{4}+\frac{1}{4}\sin 2t-\frac{t}{4}\cos 2t.$

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UNIT-IV.

VECTOR DIFFERENTIATION.

Introduction: Let S be a set of real no's. corresponding to each scalar $t \in S$, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be vector valued function.

Let $\bar{f}$ be a vector function on a interval I and $a \in I$, then $\underset{t \to a}{Lt} \frac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists is called the derivative of $\bar{f}$ at 'a' and is denoted by $f'(a) = \frac{d\bar{f}}{dt}$ at $t = a$.

Point Functions:

a) Scalar Point function: If to each pt $P(x,y,z)$ of a region R in space, a unique scalar ϕ is associated the ϕ is called scalar pt function.

Eg: The temp ϕ at any pt P of a body is a scalar point function.

b) Vector point function: If to each point $P(x,y,z)$ of a region R in a space, a unique vector $\bar{f}$ is associated the $\bar{f}$ is called vector point function.

Eg: the velocity $\bar{V}$ of a particle moving in a certain region at any time t is a vector point function.

Vector operator: The vector operator denoted by ∇ is defined by $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ is called differential operator.

Gradient:

The gradiend at a scalar point function $\phi(x,y,z)$ denoted by grad $\phi = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$ which is a vector function.

Geometrically $\nabla\phi$ denotes the normal to the surface $\phi(x,y,z) = c$.

unit normal vector $\hat{n}$ along $\nabla\phi$ is $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Directional derivative: It represents the rate of change of ϕ w.r.t distance at P in the direction of unit vector $\hat{a}$. ie., $\frac{d\phi}{ds} = \underset{\delta s \to 0}{Lt} \frac{\delta\phi}{\delta s}$.

∴ D.D of ϕ at P in direction of a unit vector $\hat{a}$ is given by $\frac{d\phi}{ds} = \nabla\phi \cdot \hat{a} = \nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|}$.

NOTE:- 1) The Maximum direction of ϕ is $|\nabla\phi|$

2) If $\bar{n}_1 = \nabla\phi$ is normal at P and

$\bar{n}_2 = \nabla\phi$ " " at Q, then angle θ bet. the normals is $\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$

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1). Find the unit normal vector at the pt. $(1,-1,2)$ to the Surface $x^2y+y^2z+z^2x=5$

Sol Let $\phi = x^2y+y^2z+z^2x-5=0$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}$$

$$= i(2xy+z^2)+j(x^2+2yz)+k(y^2+2zx)$$

∴ unit normal vector.

$$\nabla\phi|_{(1,-1,2)} = 2i-3j+5k$$

∴ unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i-3j+5k}{\sqrt{4+9+25}} = \frac{2i-3j+5k}{\sqrt{38}}$

② S.T $\nabla[f(r)] = \frac{f'(r)}{r}\cdot\bar{r}$

Sol $\bar{r} = xi+yj+zk \Rightarrow r=\sqrt{x^2+y^2+z^2} \Rightarrow r^2=x^2+y^2+z^2$

$$\Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r};\quad \frac{\partial r}{\partial y}=\frac{y}{r};\quad \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\nabla[f(r)] = \Sigma i\cdot\frac{\partial}{\partial x}f(r) = \Sigma i\cdot f'(r)\cdot\frac{\partial r}{\partial x}$$

$$= \Sigma i\cdot\left[f'(r)\cdot\frac{x}{r}\right] = \frac{f'(r)}{r}(xi+yj+zk) = \frac{f'(r)}{r}\bar{r}$$

③ Find the directional derivative of xyz^2+xz at $(1,1,1)$ in ~~a~~ direction of normal to surface $3xy^2+y=z$ at $(0,1,1)$

Sol normal to the surface $\phi = 3xy^2+y-z=0$

is $\nabla\phi = i(3y^2)+j(6xy+1)+k(-1)$

$$\nabla\phi|_{(0,1,1)} = 3i+j-k$$

unit normal vector $\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3i+j-k}{\sqrt{11}}$

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given $g(x,y,z) = xyz^2 + xz$, Then

$$\nabla g = (yz^2+z)i + xz^2 j + (2xyz + x)k$$

$$\nabla g\big|_{(1,1,1)} = 2i + j + 3k$$

$\therefore$ D.D is $\nabla g \cdot \hat{A} = (2i+j+3k) \cdot \frac{3i+j-k}{\sqrt{11}} = \frac{4}{\sqrt{11}}$

(4) Find the angle between the normals to the surface $xy = z^2$ at the pt's $(4,1,2)$ and $(3,3,-3)$

Sol normal to surface $f = xy - z^2 = 0$ is

$$\bar{n} = \nabla f = yi + xj - 2zk$$

at $(4,1,1) = \bar{n}_1 = i + 4j - 4k$

at $(3,3,-3) = \bar{n}_2 = 3i + 3j + 6k$

angle bet n_1 & n_2 is $\cos\theta = \frac{\bar{n}_1 \bar{n}_2}{|n_1||\bar{n}_2|}$

$$= \frac{3+12-24}{\sqrt{33}\,\sqrt{54}} = \frac{-9}{\sqrt{33}\,\sqrt{54}}$$

<u>Divergence of a vector</u> :- The divergence of a continuously differentiable vector point fun $\bar{f}$, denoted by

$$\operatorname{div}\bar{f} = \nabla \cdot \bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(f_1 i + f_2 j + f_3 k)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$ which is a scalar fun

<u>Solenoidal Vector</u> :- If $\nabla \cdot \bar{f} = 0$, then $\bar{f}$ is said to be solenoidal vector

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Curl of a Vector:- The curl of a continuously differentiable vector pt. fun denoted by curl $\bar{f} = \nabla \times \bar{f}$ is defined by

$$\text{curl } \bar{f} = \nabla \times \bar{f} = i \times \frac{\partial \bar{f}}{\partial x} + \bar{j}\frac{\partial \bar{f}}{\partial y} + \bar{k}\frac{\partial \bar{f}}{\partial z}$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector:- If $\nabla \times \bar{f} = 0$, then $\bar{f}$ is said to be irrotational vector.

Eg:- 1) Find div $\bar{f}$ when $\bar{f} = \text{grad}(x^3+y^3+z^3-3xyz)$

Sol Let $\phi = x^3+y^3+z^3-3xyz$

$f = \text{grad } \phi = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

$$\therefore \text{div } f = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x + 6y + 6z$$

2) Find div $\bar{f}$ where $\bar{f} = r^n \bar{r}$. Find n if $\bar{f}$ is solenoidal

Sol $\bar{r} = xi+yj+zk$; $r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r} ; \frac{\partial r}{\partial y} = \frac{y}{r} ; \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{div } \bar{f} = \Sigma \frac{\partial \bar{f_1}}{\partial x} = \Sigma \frac{\partial}{\partial x}(r^n \bar{r}) = \Sigma \frac{\partial}{\partial x} r^n x$$

$$= \Sigma \left(n r^{n-1} \frac{\partial r}{\partial x}\right) x + r^n \cdot 1$$

$$= \Sigma (n r^{n-2} x) x + r^n$$

$$= 3r^n + (x^2+y^2+z^2) n r^{n-2}$$

$$= 3r^n + n r^n = (n+3) r^n$$

if $\bar{f} = r^n \bar{r}$

$\bar{f} = r^n(xi+yj+zk)$

$\Rightarrow f_1 = r^n x$

$f_2 = r^n y$

$f_3 = r^n z$

$\text{div } \bar{f} = \Sigma \frac{\partial f_1}{\partial x}$

if $\bar{f}$ is solenoidal $(n+3)r^n = 0 \Rightarrow n = -3$

eg:- If $\bar{a}$ is a constant vector. P.T curl $\left[\frac{\bar{a}\times\bar{r}}{r^3}\right] = -\frac{\bar{a}}{r^3}+\frac{3\bar{r}}{r^5}(\bar{a}\cdot\bar{r})$

sol: we have $\bar{r} = xi+yj+zk$

$$\frac{\partial\bar{r}}{\partial x} = i \;;\; \frac{\partial\bar{r}}{\partial y} = j \;;\; \frac{\partial\bar{r}}{\partial z} = k$$

$$\text{curl}\left[\frac{\bar{a}\times\bar{r}}{r^3}\right] = \sum \bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)$$

consider $\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{a}\times\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right)$

$$= \bar{a}\times\left\{-\frac{3}{r^4}\bar{r}\frac{\partial r}{\partial x}+\frac{1}{r^3}\frac{\partial\bar{r}}{\partial x}\right\}$$

$$= \bar{a}\times\left\{-\frac{3}{r^5}x\bar{r}+\frac{i}{r^3}\right\} = \frac{\bar{a}\times i}{r^3} - \frac{3x(\bar{a}\times\bar{r})}{r^5}$$

$$i\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \bar{i}\times\left[\frac{\bar{a}\times i}{r^3} - \frac{3x}{r^5}(\bar{a}\times\bar{r})\right]$$

$$= \frac{\bar{i}\times(\bar{a}\times i)}{r^3} - \frac{3x}{r^5}\,\bar{i}\times(\bar{a}\times\bar{r})$$

$$\left[\because a\times(b\times c) = (a\cdot c)b - (a\cdot b)c\right]$$

$$= \frac{(\bar{i}\cdot\bar{i})\,\bar{a} - (\bar{i}\cdot\bar{a})i}{r^3} - \frac{3x}{r^5}\left[(\bar{i}\cdot\bar{r})\bar{a} - (\bar{i}\cdot\bar{a})\bar{r}\right]$$

$$= \frac{\bar{a} - a_1 i}{r^3} - \frac{3x}{r^5}(x\bar{a} - a_1\bar{r})$$

$$\sum\bar{i}\times\frac{\partial}{\partial x}\left(\frac{\bar{a}\times\bar{r}}{r^3}\right) = \sum\frac{\bar{a}-a_1 i}{r^3} - \frac{3}{r^5}\sum(x^2\bar{a} - a_1 x\bar{r})$$

$$= \frac{3\bar{a}-\bar{a}}{r^3} - \frac{3}{r^5}\bar{a}(r^2) + \frac{3\bar{r}}{r^5}(a_1x+a_2y+a_3z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}\cdot\bar{a})$$

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$$= -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}\cdot\bar{a})$$

Hence proved.

(2) (3) Find whether $\bar{F} = (x^2-y^3)i + (y^2-3x)j + (z^2-xy)k$ is irrotational and find scalar potential fun.

Sol:

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^3 & y^2-3x & z^2-xy \end{vmatrix} = i(-x) - j(-y) + k(-3+3y^2)$$

$$= -xi + yj + k(3y^2-3) \neq 0$$

$\therefore$ F is not irrotational $\Rightarrow$ Scalar potential does not exist

(3) (4) S.T the vector $(x^2-yz)i + (y^2-zx)j + (z^2-xy)k$ is irrotational and find scalar potential.

Sol:

$$\bar{f} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$$

$$\text{curl}\,\bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= i(-x+x) - j(-y+y) + k(-z+z) = 0$$

$\therefore$ $\bar{f}$ is irrotational

comparing components we get

$$\frac{\partial \phi}{\partial x} = x^2-yz \Rightarrow \phi = \frac{x^3}{3} - xyz + f_1(y,z)$$

$$\frac{\partial \phi}{\partial y} = y^2-zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(x,z)$$

$$\frac{\partial \phi}{\partial z} = z^2-xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y)$$

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$$\Rightarrow \phi = \frac{x^3+y^3+z^3}{3} - xyz + \text{constant}$$

is required scalar potential.

Operators :- div.(grad r^n) $= n(n+1)\, r^{n-2}$

1) P.T $\left\{ \begin{array}{l} \text{div.(grad } r^n) \text{ (or)} \\ \nabla^2(r^n) = n(n+1)\, r^{n-2} \end{array}\right.$

Sol Let $\bar{r} = xi + yj + zk \Rightarrow r^2 = x^2+y^2+z^2$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r};\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2(r^n) = \nabla\cdot\nabla(r^n)$$

$$\nabla(r^n) = \sum i\frac{\partial}{\partial x}(r^n) = \sum i\, n r^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum i\, n r^{n-1}\cdot\frac{x}{r} = \sum i\, n\, x\, r^{n-2}$$

$$\nabla^2(r^n) = \sum\frac{\partial}{\partial x}\left[n\, x\, r^{n-2}\right] = \sum n\left[x(n-2)\, r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}\right]$$

$$= \sum n\left[x^2(n-2)\, r^{n-4} + r^{n-2}\right]$$

$$= \sum n(n-2)\, x^2 r^{n-4} + n\, r^{n-2}$$

$$= n(n-2)\, r^{n-4}(x^2+y^2+z^2) + 3n\, r^{n-2}$$

$$= n(n-2)\, r^{n-2} + 3n\, r^{n-2}$$

$$= r^{n-2}\left[n^2 - 2n + 3n\right] = n(n+1)\, r^{n-2}$$

(2) (2) P.T $\nabla \times \left[\dfrac{\bar{a} \times \bar{r}}{r^n}\right] = \dfrac{(2-n)\bar{A}}{r^n} + \dfrac{n(\bar{r}\cdot\bar{A})\bar{r}}{r^{n+2}}$

Sol: $\bar{r} = xi + yj + zk \Rightarrow r^2 = x^2 + y^2 + z^2$

$\dfrac{\partial \bar{r}}{\partial x} = i$; $\dfrac{\partial \bar{r}}{\partial y} = j$; $\dfrac{\partial \bar{r}}{\partial z} = k$; $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$; $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$; $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \sum i \times \frac{\partial}{\partial x}\left(\frac{\bar{a} \times \bar{r}}{r^n}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \bar{a} \times \frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^n}\right) = \bar{a} \times \left[\frac{r^n \bar{i} - \bar{r}\, n\, r^{n-1}\cdot x/r}{r^{2n}}\right]$$

$$= \bar{a} \times \left[\frac{r^n \bar{i} - \bar{r}\, n\, x\, r^{n-2}}{r^{2n}}\right]$$

$$= \frac{\bar{a} \times \bar{i}}{r^n} - \frac{n x}{r^{n+2}}(\bar{a} \times \bar{r})$$

$$i \times \frac{\partial}{\partial x}\left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \frac{\bar{i} \times (\bar{a} \times \bar{i})}{r^n} - \frac{n x}{r^{n+2}}\, \bar{i} \times (\bar{a} \times \bar{r})$$

$$\left[\because \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}\right]$$

$$= \frac{(\bar{i}\cdot\bar{i})\bar{a} - (\bar{i}\cdot\bar{a})\bar{i}}{r^n} - \frac{n x}{r^{n+2}}\left[(\bar{i}\cdot\bar{r})\bar{a} - (\bar{i}\cdot\bar{a})\cdot\bar{r}\right]$$

$$= \frac{\bar{a} - a_1 \bar{i}}{r^n} - \frac{n x}{r^{n+2}}\left[x\bar{a} - a_1 \bar{r}\right]$$

$$\sum \bar{i} \times \frac{\partial}{\partial x}\left[\frac{\bar{a} \times \bar{r}}{r^n}\right] = \frac{3\bar{a} - (a_1\bar{i} + a_2 j + a_3 k)}{r^n}$$

$$- \frac{n}{r^{n+2}}\left[(x^2 + y^2 + z^2)\bar{a}\right]$$

$$+ \frac{n\bar{r}}{r^{n+2}}(a_1 x + a_2 y + a_3 z)$$

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$$= \frac{3\bar{a}-\bar{a}}{r^n} - \frac{n}{r^{n+2}}(r^2\bar{a}) + \frac{n\bar{r}}{r^{n+2}}(a_1x+a_2y+a_3z)$$

$$= \frac{2\bar{a}}{r^n} - \frac{n}{r^n}\bar{a} + \frac{n\bar{r}}{r^{n+2}}(\bar{a}\cdot\bar{r})$$

$$= (2-n)\frac{\bar{a}}{r^n} + \frac{n\vec{r}}{r^{n+2}}(\bar{a}\cdot\bar{r})$$

Vector Identities :-

1) If $\bar{a}$ is a differentiable fun and ϕ is a differentiable scalar fun, then $\text{div}(\phi\bar{a}) = (\text{grad}\,\phi)\cdot\bar{a} + \phi\,\text{div}\,\bar{a}$

sol

$$\text{div}(\phi\bar{a}) = \nabla\cdot(\phi\bar{a}) = \sum i\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum i\cdot\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum\left(i\frac{\partial\phi}{\partial x}\right)\cdot\bar{a} + \sum\left(i\cdot\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= (\nabla\phi)\cdot\bar{a} + (\nabla\cdot\bar{a})\phi$$

(2) P.T $\text{curl}(\phi\bar{a}) = (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$

sol

$$\text{curl}\,\phi\bar{a} = \nabla\times(\phi\bar{a}) = \sum i\times\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \sum i\times\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum\left(i\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \sum\left(i\times\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= \nabla\phi\times\bar{a} + (\nabla\times\bar{a})\phi = \text{grad}\,\phi\times\bar{a} + \text{curl}\,\bar{a}\,\phi$$

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(11)

(3) (3) P.T $grad(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times curl\,\bar{a} + \bar{a}\times curl\,\bar{b}$

sol: Consider $\bar{a}\times curl\,\bar{b} = \bar{a}\times(\nabla\times\bar{b})$

$$= \bar{a}\times\sum \bar{i}\times\frac{\partial\bar{b}}{\partial x} = \sum \bar{a}\times\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left\{\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{i} - (\bar{a}\cdot\bar{i})\frac{\partial\bar{b}}{\partial x}\right\}$$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - \left\{\bar{a}\cdot\sum \bar{i}\frac{\partial}{\partial x}\right\}\bar{b}$$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} \quad -(1)$$

11^{ly} $\bar{b}\times curl\,\bar{a} = \sum \bar{i}\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a} \quad -(2)$

(1) + (2) $\Rightarrow$ $\bar{a}\times curl\,\bar{b} + \bar{b}\times curl\,\bar{a}$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b}$$

$$+ \sum \bar{i}\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a}$$

$\Rightarrow$ $\bar{a}\times curl\,\bar{b} + \bar{b}\times curl\,\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + (\bar{b}\cdot\nabla)\bar{a}$

$$= \sum \bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) + \sum \bar{i}\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum \bar{i}\left[\bar{a}\cdot\frac{\partial\bar{b}}{\partial x} + \bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \sum \bar{i}\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b}) = \nabla(\bar{a}\cdot\bar{b})$$

(4) (4) $curl(\bar{a}\times\bar{b}) = \bar{a}\,div\,\bar{b} - \bar{b}\,div\,\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$

sol: $curl(\bar{a}\times\bar{b}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$

$$= \sum \bar{i}\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum i\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum \bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left\{(\bar{i}\cdot\bar{b})\frac{\partial\bar{a}}{\partial x} - \left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right\}$$

$$+ \sum\left\{\left(\bar{i}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (\bar{i}\cdot\bar{a})\frac{\partial\bar{b}}{\partial x}\right\}$$

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$$= \sum (\bar{b}\cdot\bar{i})\frac{\partial \bar{a}}{\partial x} - \sum\left(\bar{i}\cdot\frac{\partial \bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{i}\cdot\frac{\partial \bar{b}}{\partial x}\right)\bar{a} - \left(\bar{a}\cdot\sum \bar{i}\frac{\partial}{\partial x}\right)\bar{b}$$

$$= (\bar{b}\cdot\nabla)\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\nabla\cdot\bar{b})\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= (\nabla\cdot\bar{b})\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= \bar{a}\,\mathrm{div}\,\bar{b} - \bar{b}\,\mathrm{div}\,\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

Pb's:-

1) P.T $\mathrm{div}\{(\bar{r}\times\bar{a})\times\bar{b}\} = -2(\bar{b}\cdot\bar{a})$

$\mathrm{curl}\{(\bar{r}\times\bar{a})\times\bar{b}\} = \bar{b}\times\bar{a}$

$$\boxed{\because \mathrm{div}(\phi\bar{a}) = \bar{a}\,\mathrm{grad}\,\phi + \phi\,\mathrm{div}\,\bar{a}}$$

Sol

a) $\mathrm{div}\{(\bar{r}\times\bar{a})\times\bar{b}\} = \mathrm{div}\{(\bar{r}\cdot\bar{b})\bar{a} - (\bar{a}\cdot\bar{b})\bar{r}\}$

$= \{(\bar{r}\cdot\bar{b})\,\mathrm{div}\,\bar{a} + \bar{a}\,\mathrm{grad}(\bar{r}\cdot\bar{b})\}$

$- \{(\bar{a}\cdot\bar{b})\,\mathrm{div}\,\bar{r} + \bar{r}\cdot\mathrm{grad}(\bar{a}\cdot\bar{b})\}$

we have $\mathrm{div}\,\bar{a} = 0$; $\mathrm{div}\,\bar{r} = 3$; $\mathrm{grad}(\bar{a}\cdot\bar{b}) = 0$.

$\therefore \mathrm{div}\{(\bar{r}\times\bar{a})\times\bar{b}\} = 0 + \bar{a}\cdot\mathrm{grad}(\bar{r}\cdot\bar{b}) - 3(\bar{a}\cdot\bar{b})$

$= \bar{a}\cdot\sum \bar{i}\frac{\partial}{\partial x}(\bar{r}\cdot\bar{b}) - 3(\bar{a}\cdot\bar{b})$

$= \bar{a}\cdot\sum \bar{i}(\bar{i}\cdot\bar{b}) - 3(\bar{a}\cdot\bar{b})$

$= \bar{a}\cdot\bar{b} - 3(\bar{a}\cdot\bar{b}) = -2(\bar{a}\cdot\bar{b})$

b) $\mathrm{curl}[(\bar{r}\times\bar{a})\times\bar{b}] = \mathrm{curl}[(\bar{r}\cdot\bar{b})\bar{a} - (\bar{a}\cdot\bar{b})\bar{r}]$

$= \mathrm{curl}(\bar{r}\cdot\bar{b})\bar{a} - \mathrm{curl}(\bar{a}\cdot\bar{b})\bar{r}$

$= (\bar{r}\cdot\bar{b})\,\mathrm{curl}\,\bar{a} + \mathrm{grad}(\bar{r}\cdot\bar{b})\times\bar{a}$

$= 0 + \nabla(\bar{r}\cdot\bar{b})\bar{a}$ $[\because \mathrm{curl}\,\bar{a} = 0)$

$= \bar{b}\times\bar{a}$ $[\because \mathrm{grad}(\bar{r}\cdot\bar{b}) = \bar{b}]$

Unit-V :- Vector Integration

Chapter 1 :- Vector Integration

2 : Vector Integral Theorems.

Introduction :- The concept of a line integral is a natural generalization of the concept of a definite integral $\int_a^b f(x)\,dx$ where the integral $f(x)$ exists $\forall\ x \in [a, b]$

The concept of a surface integral is a natural generalization of double integral and the concept of volume integral is a generalization of triple integral.

Line Integrals :-

Let $\bar{r} = \bar{f}(t)$ define a smooth curve C. joining pts. A and B. Let ds be the differential of arc length at $P \in C$. Then $\frac{d\bar{r}}{ds} = \bar{t}$ is the unit vector along the tangent to the curve C at P. Let $F(r)$ be a vector pt. fun ~~and~~ defined and continuous along C. The component of $\bar{F}(r)$ along the tangent at P is $\bar{F}(r) \cdot \bar{t}$.

Then $\int \bar{F} \cdot \bar{t}\, ds$ taken along the curve C is called the line integral of F along C.

Circulation :- If $\bar{V}$ is velocity of a fluid particle and C is closed curve, then $\oint_c \bar{V} \cdot d\bar{r}$ is called circulation of $\bar{V}$ round the curve C.

If $\int_c \bar{v} \cdot d\bar{r} = 0$, then $\bar{v}$ is called conservative i.e no work is done. also $\bar{v}$ is called irrotational.

Eg: Using line integral. Find the work done by the force $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ along the lines from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ and to $(1,1,1)$

Sol: Given $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$

$\bar{r} = xi + yj + zk$; Then $d\bar{r} = dx\,i + dy\,j + dz\,k$

$\therefore \bar{F} \cdot d\bar{r} = (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$

a) along $(0,0,0)$ to $(1,0,0)$:-

$\Rightarrow x \to 0$ to 1 ; $y = 0 \Rightarrow dy = 0$
$z = 0 \Rightarrow dz = 0$

$$\therefore I_1 = \int_0^1 3x^2 dx = x^3\Big|_0^1 = 1$$

b) along $(1,0,0)$ to $(1,1,0)$:- $x = 1 \Rightarrow dx = 0$
$y \to 0$ to 1
$z = 0 \Rightarrow dz = 0$

$$I_2 = \int_0^1 0\, dy = 0$$

c) along $(1,1,0)$ to $(1,1,1)$:- $x = 1 \Rightarrow dx = 0$
$y = 1 \Rightarrow dy = 0$

$z \to 0$ to 1

$$I_3 = \int_0^1 20z^2 dz = \frac{20}{3}$$

$\therefore$ Total work $= 1 + 0 + \frac{20}{3} = \frac{23}{3}$

(2). Find the circulation of $\bar{F} = (2x-y+2z)\bar{i} + (x+y-z)\bar{j} + (3x-2y-5z)\bar{k}$ along the circle $x^2+y^2=4$ in xy plane.

sol: $\bar{F}\cdot d\bar{r} = (2x-y+2z)\,dx + (x+y-z)\,dy + (3x-2y-5z)\,dz$

In the xy plane $z=0 \Rightarrow dz=0$

$\therefore$ circulation $= \int_C \bar{F}\cdot d\bar{r}$

$$= \int_C (2x-y+\cancel{2z})\,dx + (x+y-\cancel{z})\,dy.$$

$\because$ C is $x^2+y^2=4$, taking $x = 2\cos\theta$; $y = 2\sin\theta$.

$\Rightarrow dx = -2\sin\theta\, d\theta$; $dy = 2\cos\theta\, d\theta$

$$\Rightarrow C = \int_0^{2\pi} (4\cos\theta - 2\sin\theta)(-2\sin\theta)\,d\theta + (2\cos\theta + 2\sin\theta)(2\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} -8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{2\pi} (4 - 4\sin\theta\cdot\cos\theta)\,d\theta = 8\pi$$

Surface Integrals :-

The ~~surface~~ integral which ~~~~ ~~is~~ to be evaluated over a surface integral is called Surface integral.

Suppose S is a surface of finite area and $f(x,y,z)$ is a single valued fun defined over Surface S.

Sub divide the area S into n elementary areas $\delta s_1, \delta s_2 \ldots \delta s_n$ choose an arbitrary pt P_i whose coordinates are (x_i, y_i, z_i).

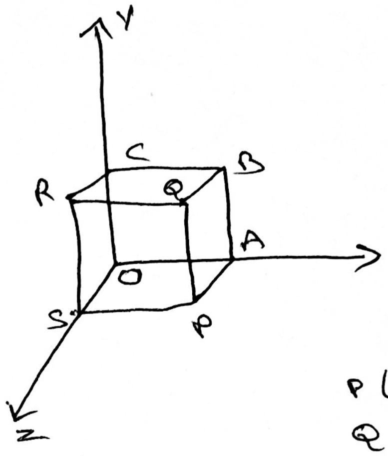
Consider the sum $\sum_{i=1}^{n} f(P_i) \cdot \delta S_i$

The limit of this sum as $n \to \infty$ & $\delta S_i \to 0$, if exists called surface integral of $f(x,y,z)$ over S is denoted by $\iint f(x,y,z)\, ds$

Eg:- If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$, evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$ where S is the surface of the cube bounded by $x=0, x=a$, $y=0$; $y=a$; $z=0$; $z=a$.

Sol consider cube.

here $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$



A(a,0,0)
B(a,a,0)
C(0,a,0)
O(0,0,0)

P(a,0,a)
Q(a,a,a)
R(0,a,a)
S(0,0,a)

For PQRS:-

$x \to 0$ to a

$y \to 0$ to a

$z = a \Rightarrow dz = 0$

$\bar{n} = \bar{k}$

$\therefore \bar{F}\cdot\bar{n} = yz = ay$

$\therefore \iint \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{a}\int_{y=0}^{a} ay\, dx\, dy = \frac{a^3}{2}\int_0^a dx = \frac{a^4}{2}$

For ABCO:-

$z=0$; $x \to 0$ to a

$y \to 0$ to a

$\bar{n} = -\bar{k}$

$\therefore \bar{F}\cdot\bar{n} = -yz = -ay = 0$

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For PABQ :-

$x = a$; $y \rightarrow 0$ to a
$z \rightarrow 0$ to a

$\bar{n} = \bar{i}$

$\therefore \bar{F} \cdot \bar{n} = 4xz = 4az$

$$\therefore \iint \bar{F} \cdot \bar{n} \, ds = \int_{y=0}^{a} \int_{z=0}^{a} 4az \, dy \, dz$$

$$= 2a^4$$

For RCOS :- $x = 0$; $y \rightarrow 0$ to a
$z \rightarrow 0$ to a

$\bar{n} = -\bar{i}$

$\bar{F} \cdot \bar{n} = 0$

$\therefore \iint \bar{F} \cdot \bar{n} \, ds = 0$

For RCBQ :-

$y = a$; $x \rightarrow 0$ to a $\quad \bar{F} \cdot \bar{n} = -y^2 = -a^2$
$z \rightarrow 0$ to a

$\bar{n} = \bar{j}$

$$\therefore \iint \bar{F} \cdot \bar{n} \, ds = \int_0^a \int_0^a -a^2 \, dx \, dz = -a^4$$

For OAPS :-

$y = 0$; $x \rightarrow 0$ to a
$z \rightarrow 0$ to a

$\bar{n} = -\bar{j}$

$\bar{F} \cdot \bar{n} = 0$

$\therefore \iint \bar{F} \cdot \bar{n} \, ds = 0$

$$\therefore \iint \bar{F} \cdot \bar{n} \, ds = 2a^4 + \frac{a^4}{4} - a^4 = \frac{3a^4}{2}$$

Volume Integral :-

Suppose V is a volume bounded by a Surface S. Suppose $f(x,y,z)$ is a single valued fun defined over V. Divide the volume V into elementary volume $\delta V_1, \delta V_2 \ldots \delta V_n$.

Let $P_i(x_i, y_i, z_i)$ be an arbitrary pt. in the elementary volume δV_i.

Consider the sum $\sum_{i=1}^{n} f(P_i) \cdot \delta V_i$

The limit of this sum, if exists as $n \to \infty$ & $\delta V_i \to 0$ over V given by is called volume integral of f

$$\iiint_V f(x,y,z)\, dV$$

Eg:- If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate

a) $\int_V \nabla \cdot \bar{F}\, dV$ and $\int_V \nabla \times \bar{F}\, dV$, where V is the closed region bounded by $x=0, y=0, z=0$, $2x+2y+z=4$

sol:
$$\nabla \cdot \bar{F}\, dV = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= 4x - 2x + 0 = 2x.$$

here $z \to 0$ to $4-2x-2y$

$y \to 0$ to $4-2x$

$x \to 0$ to 2

$$\therefore \int_V \nabla \cdot \bar{F}\, dV = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{4-2x-2y} 2x\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{4-2x} 2x(4-2x-2y)\, dy\, dx$$

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$$= 4\int_0^2 \left[2xy - x^2y - \frac{xy^2}{2}\right]_0^{2-x} du$$

$$= 4\int_0^2 \left[(2x-x^2)(2-x) - \frac{x}{2}(2-x)^2\right] dx$$

$$= 4\int_0^2 \frac{2-x}{2}\left[(4x-2x^2) - (2x-x^2)\right] dx$$

$$= \int_0^2 (x^3 - 4x^2 + 4x)\, dx$$

$$= 2\left[\frac{x^4}{4} - 4\cdot\frac{x^3}{3} + 4\cdot\frac{x^2}{2}\right]_0^2 = \frac{8}{3}$$

ii) $$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$$= \bar{j} - 2y\bar{k}$$

$$\therefore \int (\nabla \times \bar{F})\, dv = \iiint_V (\bar{j} - 2yk)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (j - 2yk)\, z \Big|_0^{4-2x-2y} dx\, dy$$

$$= \int_0^2\int_0^{2-x} (j-2yk)(4-2x-2y)\, dx\, dy$$

$$= \int_0^2\int_0^{2-x} j\left[(4-2x)-2y\right] - \bar{k}\left[(4-2x)\cdot 2y - 4y^2\right] dx\, dy$$

$$= \int_0^2 \bar{j}\, (4-2x)\, y - y^2 \Big|_0^{2-x} dx$$

$$- k\int_0^2 (4-2x)\, y^2 - \frac{4y^3}{3}\Big|_0^{2-x} \cdot dx$$

$$= j\left(\int_0^2 (2-x)^2 dx - k\int_0^2 \frac{2}{3}\cdot(2-x)^3 dx\right.$$

$$= j\left[\frac{(2-x)^3}{-3}\right]\Big|_0^2 - \frac{2k}{3}\left[\frac{(2-x)}{-4}\right]_0^2$$

$$= \frac{8}{3}(j - \bar{k})$$

4

Vector Integral theorems:

Gauss - Divergence theorem: (Transformation b/w Surface Integral to volume Integral).

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then $\int_V \operatorname{div} \bar{F}\, dv = \int_S \bar{F}\cdot\bar{n}\, ds$.

Eg: verify divergence theorem for $\bar{F} = x^2\bar{i} + y^2 j + z^2 k$ ~~by~~ over the surface S of the solid cut off by the plane $x+y+z=a$, in the first octant.

Sol. Let $\phi = x+y+z-a$ be the given plane.

then grad $\phi = i+j+k$

unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{i+j+k}{\sqrt{3}}$

Let R be projection of S on ~~the~~ xy-plane.

then $y \to 0$ to $a-x$

$x \to 0$ to a

$\therefore \int_S \bar{F}\cdot\bar{n}\, ds = \iint_R \dfrac{\bar{F}\cdot\bar{n}}{|\bar{n}\cdot\bar{k}|}\, dx\, dy$

$$= \int_0^a \int_0^{a-x} \frac{x^2+y^2+z^2}{\sqrt{3}} \times \sqrt{3}\, dx\, dy.$$

$$= \int_0^a \int_0^{a-x} \left[x^2+y^2+(a-x-y)^2\right] dx\, dy.$$

$$= \int_0^a \int_0^a (2x^2+2y^2-2ax+2xy-2ay+a^2)\, dx\, dy.$$

$$= \int_0^a \left[2x^2y + \frac{xy^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x} dx$$

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$$= \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right) dx = \frac{a^4}{4}.$$

R.H.S:-

$$\operatorname{div} \bar{F} = 2(x+y+z)$$

$$\therefore \int_V \operatorname{div} \bar{F}\, dv = 2 \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)\, dx\, dy\, dz$$

$$= 2 \int_0^a \int_0^{a-x} z(x+y) + \frac{z^2}{2} \Big|_0^{a-x-y} dx\, dy$$

$$= 2 \int_0^a \int_0^{a-x} (a-x-y)\left[x+y+\frac{a-x-y}{2}\right] dx\, dy$$

$$= \int_0^a \int_0^{a-x} (a-x-y)(a+x+y)\, dx\, dy$$

$$= \int_0^a \int_0^{a-x} (a^2-x^2-y^2-2xy)\, dx\, dy$$

$$= \int_0^a \left[a^2y - x^2y - \frac{y^3}{3} - xy^2\right]_0^{a-x} dx$$

$$= \int_0^a (a-x)(2a^2-x^2-ax)\, dx = \frac{a^4}{4}$$

Hence verified.

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(2) Use Divergence theorem $\iint (xi+yj+z^2k)\,\bar{n}\,ds$ where s is the surface bounded by cone $x^2+y^2=z^2$ in the plane $z=4$.

Sol. Let $\bar{F} = xi+yj+z^2k$

By Gauss divergence theorem.

$$\iint (xi+yj+z^2k)\,\bar{n}\,ds = \int_V \nabla\cdot\bar{F}\,dv.$$

Now $\nabla\cdot\bar{F} = 1+1+2z = 2z+2$

on cone $x^2+y^2=z^2$ and $z=4 \Rightarrow x^2+y^2=16$

$z \to 0$ to 4

$y \to 0$ to $\sqrt{16-x^2}$

$x \to 0$ to 4

$$\therefore \int_V \nabla\cdot\bar{F}\,dv = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^4 2(1+z)\,dx\,dy\,dz$$

$$= \int_0^4 \int_0^{\sqrt{16-x^2}} 2\times 12\,dy\,dx = 24\int_0^4 \sqrt{16-x^2}\,dx$$

$$= 24\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4$$

$$= 24\left[8\times\frac{\pi}{2}\right] = 96\pi$$

Green's Theorem: (transformation bet line integral & double Integral).

If R is a closed region in xy-plane bounded by a simple closed curve c and if M and N are continuous function of x and y having continuous derivatives in R. Then,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

1) verify Green Theorem for $\oint (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where C is the region bounded by $y=\sqrt{x}$ and $y=x^2$

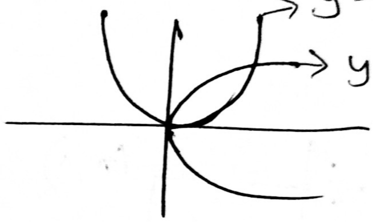
Sol Let $M=3x^2-8y^2$ and $N=4y-6xy$, then

$$\frac{\partial M}{\partial y} = -16y \quad ; \quad \frac{\partial N}{\partial x} = -6y$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (16y-6y)\,dx\,dy$.

$$= 10\iint_y y\,dx\,dy = \frac{10}{2}\int_0^1 (x-x^4)\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3}{2}$$



R.H.S :- $I = I_1 + I_2$

$I_1 \rightarrow$ line integral along $y=x^2$

$$I_1 = \int_0^1 [3x^2-(8x^2)^2]\,dx + [4x^2-6x(x^2)]\,2x\,dx.$$

$$= \int_0^1 (3x^2+8x^3-20x^4)\,dx = -1$$

$$I_2 = \int_1^0 (3x^2-8x)\,dx + (4\sqrt{x}-6x^{3/2})\frac{1}{2\sqrt{x}}\,dx$$

$$= \int_0^1 (3x^2-11x+2) = \frac{5}{2}$$

$$\therefore I_1 = I_1 + I_2 = -1 + \frac{5}{2} = 3/2.$$

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(2) Evaluate by Green Theorem $\oint_C (x^2-\cosh y)\,dx + (y+\sin x)\,dy$ where C is the rectangle with vertices $(0,0)\ (\pi,0)\ (\pi,1)\ (0,1)$.

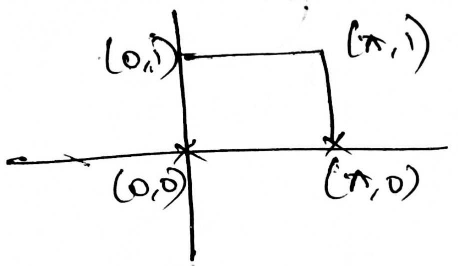
Sol. $M = x^2-\cosh y$; $N = y+\sin x$

$\therefore \frac{\partial M}{\partial y} = -\sinh y$; $\frac{\partial N}{\partial x} = \cos x$

By Green's theorem $\oint_C M\,dx + N\,dy = \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\Rightarrow \oint_C (x^2-\cosh y)\,dx + (y+\sin x)\,dy$$

$$= \iint_R (\cos x + \sinh y)\,dx\,dy$$



$$= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_0^{\pi} [\cos x + \cosh 1 - 1]\,dx$$

$$= [\sin x + x\cosh 1 - x]_0^{\pi}$$

$$= \pi(\cosh 1 - 1)$$

<u>Stokes</u> theorem :- (Transformation <u>bet</u> Line Integral and surface Integral).

Let S be a open surface bounded by a closed, non-intersecting curve C. If $\bar{F}$ is any differentiable vector point function then $\oint_C \bar{F}\cdot d\bar{r}$

$$= \int_S \text{curl}\ \bar{F}\cdot\bar{n}\,ds.$$

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Eg 1) Verify Stokes Th for $\bar{F} = -y^3\bar{i} + x^3\bar{j}$ where S is the circular disc $x^2+y^2 \le 1$; $z=0$.

Sol Given that $\bar{F} = -y^3\bar{i} + x^3\bar{j}$

$\because$ $x^2+y^2=1$; $z=0$

using $x=\cos\theta$; $y=\sin\theta$; $z=0$

$\theta \to 0$ to 2π

$dx = -\sin\theta\, d\theta$; $dy = \cos\theta\, d\theta$.

L.H.S

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_C -y^3 dx + x^3 dy$$

$$= \int_0^{2\pi} [\cos^4\theta + \sin^4\theta]\, d\theta$$

$$= \int_0^{2\pi} (1 - 2\sin^2\theta\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} d\theta - 2\int_0^{2\pi} \sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= \theta\Big|_0^{2\pi} - \frac{2}{4}\int_0^{\pi} \sin^2 2\theta\, d\theta$$

$$= 2\pi - \frac{1}{4}\int_0^{2\pi} (1-\cos 4\theta)\, d\theta$$

$$= 2\pi - \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= 2\pi - \frac{1}{4}\left[2\pi - \frac{1}{4}\times 0 - 0 + 0\right]$$

$$= 2\pi - \frac{2\pi}{4} = \frac{3\pi}{2}$$

R.H.S :-

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{K}(3x^2+3y^2$$

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$\therefore \int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3\int_S (x^2+y^2)\, \bar{k} \cdot \bar{n}\, ds$

we have $(\bar{k} \cdot \bar{n})\, ds = dx\,dy$ and R is region of xy-plane.

$\therefore \iint (\nabla \times \bar{F}) \cdot \bar{n}\, ds = 3\int_S (x^2+y^2)\, \bar{k} \cdot \bar{n}\, ds$

we have $(\bar{k} \cdot \bar{n})\, ds = dx\,dy$

$= 3\iint_R x^2+y^2\, dx\, dy$

put $x = r\cos\theta$; $y = r\sin\theta \Rightarrow dx\,dy = r\,dr\,d\theta$

$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2 \cdot r\, dr\, d\theta = 3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta$

$= \frac{3}{4} \times 2\pi = \frac{3}{2}\pi$

$\therefore$ L.H.S = R.H.S

(2) Apply stokes Th to evaluate $\oint_C y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of the sphere $x^2+y^2+z^2 = a^2$ and $x+z = a$ and

Sol The intersection of the sphere $x^2+y^2+z^2 = a^2$ and the plane $x+z = a$ is a circle in the plane $x+z = a$; with AB as diameter.

Eq of the plane is $x+z = a \Rightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA = OB = a$ i.e $A(a,0,0)$ and $B(0,0,a)$

length of diameter, $AB = \sqrt{a^2+a^2+0} = a\sqrt{2}$

radius of the circle $r = \frac{a}{\sqrt{2}}$

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Let $\bar{F}\cdot d\bar{r} = y\,dx + z\,dy + x\,dz$

$= (y i + z j + x k)\cdot(i\,dx + j\,dy + k\,dz)$

$\therefore \operatorname{curl}\bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(i+j+k)$

Let $\bar{n} = \frac{\nabla S}{|\nabla S|}$ here $S = x + z - a$

$\nabla S = i + k$

$\bar{n} = \frac{i+k}{\sqrt{2}}$

hence $\oint_c \bar{F}\cdot d\bar{r} = \int(\nabla\times F)\cdot\bar{n}\,ds$

$= -\int(i+j+k)\cdot\left(\frac{i+k}{\sqrt{2}}\right)ds = \int\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,ds$

$= -\sqrt{2}\int_s ds = -\sqrt{2}\,S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right)$

$= \frac{\pi a^2}{\sqrt{2}}$

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